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Ph.D. Thesis

Orthogonal polynomials with respect to differential operators and matrix orthogonal polynomials

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A mi madre, a los presentes y a los ausentes

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"Superstition brings bad luck." - Raymond Smullyan.
"The limits of language mean the limits of my world"- Ludwig Wittgenstein.

Resumen y aportaciones

En esta tesis concierne con el concepto de polinomios ortogonales con respecto a un operador diferencial, el estudio del comportamiento asintótico fuerte de las funciones propias polinomiales de operadores exactamente solubles y polinomios ortogonales matriciales. El trabajo está dividido en siete capítulos.

En el Capítulo 1 presentamos algunos conceptos de la teoría general de polinomios ortogonales, así como el estado del arte de la teoría que preceden a los resultados de esta tesis.

El objetivo del Capítulo 2 es el estudio de los polinomios ortogonales con respecto a un operador de Jacobi

$$\mathcal{L}^{(\alpha,\beta)}[f] = (1-x^2)(x)f''(x) + (\beta-\alpha-(\alpha+\beta+2)x)f'(x), \alpha, \beta > -1, f \in \mathbb{P}$$

y una medida finita positiva de Borel μ soportada en $[-1,1]$ la cual satisface ciertas condiciones. Para un número entero positivo m, estudiamos las condiciones sobre la medida μ para garantizar la existencia de una sucesión infinita de polinomios mónicos $\{Q_n\}_{n=m+1}^{\infty}$, $deg[Q_n] = n$, que satisfacen la condición

$$\int \mathcal{L}^{(\alpha,\beta)}[Q_n](x)x^k d\mu(x) = 0 \quad \text{para todo} \quad 0 \leq k \leq n-1.$$

Se estudian las propiedades algebráicas y analíticas de esta sucesión y se muestra un modelo de dinámica de fluídos para la interpretación de los ceros de estos polinomios.

En el Capítulo 3, abordaremos el caso de ortogonalidad respecto a un operador de Laguerre o Hermite. Probaremos la existencia de relaciones de recurrencia para polinomios ortogonales con respecto a estas clase de operadores así como para las derivadas de estos polinomios. Como en el caso del operador de Jacobi considerado en Capítulo 2, para los ceros de los polinomios y los ceros de los derivadas es posible encontrar un modelo de dinámica de los fluidos. También estudiamos propiedades asintóticas de estos polinomios, escalando con un parámetro adecuado.

En el Capítulo 4 se generalizan los resultados de los Capítulos 2 y 3. Se estudian los polinomios ortogonales con respecto a un operador diferencial lineal

$$\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k},$$

donde $\{\rho_k\}_{k=0}^{M}$ son polinomios tales que $deg[\rho_k] \leq k, 0 \leq k \leq M$, con igualdad para al menos un índice k. Analizaremos la unicidad de la sucesión de los polinomios así como su localización de ceros. Un fenómeno interesante que ocurre en este tipo de ortogonalidad es la existencia de operadores para los cuales la sucesión asociada de polinomios ortogonales se reduce a un conjunto finito. Para un operador dado, se encuentra una clasificación, en términos de un sistema de ecuaciones en diferencias con coeficientes variables, de las medidas para las cuales es posible garantizar la existencia de una sucesión infinita de polinomios ortogonales. También obtenemos una curva que contiene el conjunto de puntos de acumulación de los ceros de los polinomios para el caso de un operador diferencial de primer orden, dando también una fórmula para el comportamiento asintótico fuerte.

En el Capítulo 5 estudiamos el comportamiento asintótico fuerte de las funciones propias polinomiales de operadores exactamente solubles $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k}$. Las formulas que determinan el comportamiento asintótico de este tipo han atraído siempre una gran atención en relación con problemas de la teoría de polinomios ortogonales y teoría de la aproximación. Algunas propiedades de las funciones polinomiales de esta clase de operadores han sido estudiados previamente en [132] para operadores de la forma $\mathcal{L}^{(M)}[f](z) =$

$\dfrac{d^M}{dz^M}(\rho_M(z)f(z))$, donde ρ_M es un polinomio fijo de grado M y para operadores exactamente solubles por [15], [16] y [17]. Considerando la hipóptesis de que el polinomio ρ_M es real, se obtiene una fórmula para el comportamiento asintótico fuerte de las funciones propias polinomiales de $\mathcal{L}^{(M)}$ en determinados subconjuntos compactos de $\mathbb{C}$.

Como una aplicación, se considera la sucesión de polinomios ortogonales mónicos con respecto al producto interno de Sobolev,

$$\langle P, Q\rangle = P(1)\overline{Q}(1) + \mu\, P'(1)\overline{Q}'(1) + \int_{-1}^{1} P'\overline{Q}'\,dx, \quad P, Q \in \mathbb{P}, \quad \mu > 0$$

los cuales son funciones propias del operador diferencial cuarto orden, cf. [89]

$$\mathcal{L}^{(M)}[u] = (z^2-1)^2u^{(4)} + 4z(z^2-1)u^{(3)} + 2(z-1)((1+2\,A)z + 2\,A + 3)u^{(2)},$$

y se obtiene el comportamiento asintótico fuerte de la sucesión para subconjuntos compactos de $\mathbb{C}\setminus[-1,1]$.

El Capítulo 6 concierne con polinomios ortogonales matriciales. Se encuentra una clase de polinomios ortogonales matriciales de orden $N\times N$, siendo N un número natural arbitrario, los cuales son solución de una ecuación diferencial de segundo orden con coeficientes matriciales . Para matrices de tamaño $N=2$, se muestra la expresión explícita de la sucesión de polinomios ortonormales con respecto a cierto peso matricial W utilizando una fórmula de Rodrigues que estos polinomios satisfacen. En particular, se muestra que uno de los coeficientes de recurrencia para una sucesión de polinomios ortonormal asintóticamente no se comporta como un múltiplo escalar de la identidad, como ocurre en los ejemplos estudiados hasta ahora en la literatura.

En el último capitulo, se presenta un breve resúmen, conclusiones y algunos problemas abiertos.

Todos los resultados de los Capítulos 2,3,4 y 5 son novedosos y han sido enviados para considerarse como publicación a revistas incluídas en el Journal of Citations Report®, ver [23, 24, 25, 22]. Los resultados del Capítulo 6 también son novedosos y han sido publicados en [21].

Contents

Chapter 1

Introduction

1.1 On orthogonal polynomials systems

This introductory chapter deals with some basic definitions and facts of the general theory of orthogonal polynomials as well as the state of the art of the problems that we will study in the sequel.

Let $\mathbb{P}$ be the vector space of all polynomials with complex coefficients, $\mathbb{P}_n$ the vector subspace of all polynomials of degree at most n and assume that we introduce an inner product in this space. Orthogonal polynomials with respect to an inner product are formally defined as follows.

DEFINITION 1.1. *Let $\langle\cdot,\cdot\rangle : \mathbb{P}\times\mathbb{P}\to\mathbb{C}$ an inner product over $\mathbb{P}$. We will say that $\{P_n\}_{n=0}^{\infty}$ is a system of orthogonal polynomials with respect to the inner product $\langle\cdot,\cdot\rangle$ if*

- $deg[P_n]=n, \quad \forall n,$
- $\langle P_n, P_m\rangle = \delta_{n,m} d_n, \quad d_n \neq 0,$

where $deg[P_n]$ denotes the degree of the polynomial P_n and $\delta_{n,m}$ is the Kronecker delta. If

$$\|P_n\|^2 = \langle P_n, P_n\rangle = 1,$$

we will say that $\{P_n\}_{n=0}^{\infty}$ is the sequence of orthonormal polynomials with respect to the measure μ and if $P_n(x) = x^n + \cdots$ then we will say that $\{P_n\}_{n=0}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle$.

Using the Gram-Schmidt process [32], it is not difficult to see that there exists an unique family of monic orthogonal polynomials.

An important role plays inside the theory of orthogonal polynomials the multiplication operator defined as

$$T_x(p) = xp, \quad \forall p\in\mathbb{P},$$

and the fact that this operator is self adjoint with respect to the inner product considered can be used to describe an important class of orthogonal polynomials, *the standard sequences of orthogonal polynomials*.

DEFINITION 1.2. *Let $\langle\cdot,\cdot\rangle : \mathbb{P}\times\mathbb{P}\to\mathbb{C}$ an inner product over $\mathbb{P}$. We will say that $\{P_n\}_{n=0}^{\infty}$ is a standard sequence of monic orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle$ if*

$$\langle T_x(p), q\rangle = \langle p, T_x(q)\rangle.$$

The standard theory of orthogonal has been widely studied during the XX century, we refer here the classical monographs [163, 65, 70, 34], also [137, 169, 162]. From a more historical point of view, see [138, 106, 105].

Let μ be a finite positive Borel measure with support (denoted thorough all this thesis by $supp(.)$) consisting of a infinite number of points of the real line. If $supp(\mu)$ is unbounded it is assumed additionally that the moments, defined as $\int x^n d\mu(x)$, exist $\forall n \in \mathbb{Z}_+$. The most simple example of a standard inner product can be found in the space $L^2(\mu)$ of square integrable functions with respect to a measure μ with support on a subset of $\mathbb{R}$. This space has a natural structure of Hilbert space with inner product given by

$$\langle f, g \rangle = \int f(x)g(x)d\mu(x), \quad f, g \in L^2(\mu). \tag{1.1}$$

Several areas of mathematics such as continued fractions, Gaussian quadratures, moment problems, lead to consider orthogonal polynomials in $L^2(\mu)$ and the fact that these systems of functions are complete in $L^2(\mu)$ and easy to handle numerically are some reasons of its importance. In this case, the sequence of orthogonal polynomials with respect to (1.1) it is referred as to the sequence of orthogonal polynomials with respect to the measure μ.

The general theory of orthogonal polynomials really started with the investigations of Tchebychev and Stieltjes. Stieltjes work has already been discussed by Cosserat [36] shortly after Stieltjes death in 1894, we also mention the Van Assche's paper [170], on the value of the investigations by Stieltjes a century later and the Brezinski's book on the history of continued fractions [28, 10, Ch. 5, Sect. 5.2.4 on pp. 224–235] where Stieltjes work on continued fractions is shown in its historic context.

The impact of the work of Tchebychev and his student Markov has already been described by Krein in [96]. A few particular orthogonal polynomials were known before Tchebychev. Legendre and Laplace had encountered the Legendre polynomials in their work on celestial mechanics in the late eighteenth century. Laplace had found and studied the Hermite polynomials in the course of his discoveries in probability theory at the end of the eighteenth century. Other isolated instances of orthogonal polynomials occurring in the work of various mathematicians are mentioned later. It was Tchebychev who saw the possibility of a general theory and its applications. His work arose out of the theory of least squares approximation and probability; he applied his results to interpolation, approximate quadrature and other areas. He discovered the discrete analogue of the Jacobi polynomials but their importance was not recognized until centuries XX and XXI. They were rediscovered by Hahn and named after him upon their rediscovery. Geronimous has pointed out that in his first paper on orthogonal polynomials, [69], Tchebychev already had the Christoffel-Darboux formula.

Orthogonality with respect to an inner product leads to a natural generalization. A moment functional σ is a linear mapping $\sigma : \mathbb{P} \to \mathbb{C}$. For $p \in \mathbb{P}$, we write $\langle \sigma, p \rangle$ instead of $\sigma(p)$. For each $n \in \mathbb{N} \bigcup \{0\}$, where $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers; the real number $\sigma_n := \langle \sigma, x^n \rangle$ is called the n–th moment of σ. It is well known from Boas'moment theorem (see [34, p. 74]) that σ has a representation of the form

$$\langle \sigma, p \rangle = \int p d\mu, \tag{1.2}$$

where $\mu = \mu_1 + \imath\mu_2$ and μ_1, μ_2 are a finite (possibly signed) Borel measures generated from the functions $\widehat{\mu}_1 : \mathbb{R} \to \mathbb{R}, \widehat{\mu}_2 : \mathbb{R} \to \mathbb{R}$ of local bounded variation. A more recent result of [41] yields a different representation

$$\langle \sigma, p \rangle = \int p(x)w(x)dx,$$

where $w = w_1 + \imath w_2$ and w_1, w_2 are functions in the Schwartz class.

Since the publishing of [62], much progress has been made on the calculus of moment functionals, a study initiated by Maroni [123] and further advanced by the Korean school under the leadership of K.H.Kwon. For example, the moment functional σ', the derivative of σ, and $\phi\sigma$, the multiplication of σ times a polynomial ϕ are defined to be moment functionals through the formulas

$$\begin{aligned} \langle \sigma', \psi \rangle &= -\langle \sigma, \psi' \rangle, \\ \langle \phi\sigma, \psi \rangle &= \langle \sigma, \phi\psi \rangle, \end{aligned} \tag{1.3}$$

for all $\psi \in \mathbb{P}$.

Then, if there is a sequence of monic polynomials $\{P_n\}_{n=0}^{\infty}$ satisfying the condition

$$\sigma(P_n \overline{P}_m) = \delta_{n,m} d_n, \quad d_n \neq 0, \quad \forall n \in \mathbb{Z}_+,$$

we call it the sequence of monic orthogonal polynomials with respect to σ (or also with respect to μ, if μ represents σ).

Classical orthogonal polynomials are the most studied families of orthogonal polynomials. In this group we have the well known families of the Jacobi, Laguerre and Hermite (including the specials cases Legendre, Tchebychev and Gegenbauer) and Bessel. These families are also known as polynomials of hypergeometric type, indeed, they are solutions of the hypergeometric differential equation

$$A(x)y'' + B(x)y' + \lambda y = 0, \tag{1.4}$$

where A, B are fixed polynomials of degrees at most 2 and exactly 1 respectively and λ is a spectral parameter. These family are can be formally defined by means of a *Rodrigues' formula*, cf. [164, 37, 139]

$$y_n(x) = \frac{k_n}{\rho(x)}[A^n(x)\rho(x)]^{(n)},$$

where k_n is a normalizing constant and ρ satisfies the Pearson's equation

$$[A(x)\rho(x)]' = B(x)\rho(x),$$

together with some boundary conditions. There are several properties characterizing such families and can be used to define the classical OPS. The oldest one is the so called Hahn [84] characterization, unless this was firstly observed and proved for the Jacobi, Laguerre, and Hermite polynomials by Sonin in 1887. For other characterizations the reader can consult [1, 2, 34, 109, 115]. These families have important properties inside the branch of the mathematical analysis and some of them will be used thorough this work.

One of the most important problems of mathematical physics was the problem (coming from the 19th century) of proving the well-posedness of the Dirichlet and Neumann problems for the Laplace equation and more general elliptic partial differential equations of second order, so that the proof must be based on the fact that the solutions to these problems are minimizers of the Dirichlet integral. This problem attracted the attention of outstanding scientists of that time: D. Hilbert, K.O. Friedrichs, R. Courant, G. Weyl. S. Sobolev overcame the central difficulty in this problem: he found adequate function spaces, known now as *Sobolev spaces* $W^{l,p}(\Omega)$, where $p > 1, l = 0, 1, \ldots, \Omega$ is a domain in $\mathbb{R}^n$. The Sobolev space $W^{l,p}(\Omega)$ is defined as the space of functions in $L^1(\Omega)$ whose distributional derivatives of order up to l exist and belong to $L^p(\Omega)$.

In his celebrated paper [161] of 1938, Sobolev also proved the first embedding theorems (or Sobolev inequalities) which established relations between $W^{l,p}(\Omega)$ and the spaces $L^p(\Omega), C^m(\Omega)$. Sobolev spaces and Sobolev inequalities have played a fundamental role in the further development of the theory of partial differential equations, mathematical physics, differential geometry, and various fields of mathematical analysis.

Suppose that $\{\mu_j\}_{j=0}^{k}$ is a set of finite positive Borel measures supported on some subsets $\{A_j\}_{j=0}^{k}$ of the complex plane such that $supp(\mu_j)$ has an infinite number of points for at least one index and denote $\mu = (\mu_0, \ldots, \mu_k)$. The weighted Sobolev spaces $W^{k,p}(\mu)$ are a natural framework for several applications.

They are defined as the space of functions in $L^1(\mu_0)$ whose distributional derivative j exist and belong to $L^p(\mu_j)$, for all $j = 1, \ldots, k$. Thus, the classical Sobolev space $W^{k,p}(\Omega)$ is a particular case of these spaces. Of special interest is the space $H^k(\mu) = W^{k,2}(\mu)$. This space has a natural structure of inner product

$$\langle f, g\rangle \quad = \quad \sum_{j=0}^{k} \int f^{(j)}(x)\overline{g}^{(j)}(x)d\mu_j(x), \quad f, g \in H^k(\mu). \tag{1.5}$$

There are several motivations for the study of orthogonal polynomials with this kind of orthogonality, perhaps the most natural one is smooth data fitting, [100] is the first work where this problem is studied. The Spanish school around F. Marcellán, G. López and A. Martinez-Finkelshtein has been particularly active in developing this area (see the surveys [108, 125, 128] and the references therein).

In general terms, when referred to Sobolev orthogonality it is assumed that the underlying inner product involves derivatives (in the classical or distributional sense). Here we mention two cases, the diagonal case, which corresponds to the inner product defined in (1.5) and a slightly more general situation, the non diagonal case:

$$\langle \mathbf{f}, \mathbf{g}\rangle \quad = \quad \int \mathbf{f} A \mathbf{g}^*, \quad f, g \in H^k(\mu), \tag{1.6}$$

where A is an $(k+1) \times (k+1)$ Hermitian matrix with finite positive Borel measures as entries, $\mathbf{f} = (f, f', \ldots, f^{(k)})$ and $\mathbf{f}^*$ is the transpose conjugate of $\mathbf{f}$.

Either (1.5) or (1.6) define an inner product in the linear space $\mathbb{P}$ of polynomials with complex coefficients. The Gram-Schmidt process applied to the canonical basis of $\mathbb{P}$ generates the monic sequence of polynomials $\{P_n\}_{n=0}^{\infty}$, $deg[P_n] = n$. As usual, we will call these polynomials *Sobolev orthogonal polynomials*.

A related definition to Sobolev orthogonality is the *orthogonality with respect to a differential operator* given in [8], and this is one of the main objects of study of this thesis. We shall deal with this type of orthogonality in Section 1.3.

During the 1990s a very active research on Sobolev orthogonal polynomials was in progress. For a historical review of this period the reader is referred to [126, 133]. For recent applications of Sobolev polynomials see [107].

Most of the arguments for the standard theory fail in this case, e.g. it is no longer true that the zeros lie in the convex hull of the support of the measures $\mu_k, k = 0, 1, \ldots, r$. The key fact is that we have now a non standard inner product, i.e. the multiplication operator is not self adjoint. This was already noted in [7] by showing that with the following inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx + 10\int_{-1}^{0} f'(x)g'(x)dx + \int_{0}^{1} f'(x)g'(x)dx,$$

the monic Sobolev polynomial of degree 2 is

$$P_2(x) = x^2 + \frac{27}{35}x - \frac{1}{3},$$

and has a zero at $x = -1.08 \notin (-1, 1)$. In fact, the existence of zeros of Sobolev orthogonal polynomials out of the support of the measures is a frequently occurring phenomenon.

It is not even known if the zeros are bounded if all the measures $\{\mu_j\}_{j=0}^{k}$ have compact support. The most general results concerning this aspect can be found in [103, 146, 149]

Another outstanding property of this type of orthogonality is the recurrence relation. If the inner product is standard, it is not difficult to prove that the corresponding sequence of monic orthogonal polynomials satisfies a three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \quad b_n > 0.$$

In the Sobolev case the number of terms, in general, grows with the degree of the polynomials. With respect to this, in [61], the authors proved that in order to the number of terms in a recurrence relation satisfied by a sequence of orthogonal polynomials with respect to a Sobolev inner product does not depend on the degree of the polynomial it is necessary and sufficient that the measures associated to the derivatives that define the inner product must be atomic measures with a finite combination of Dirac deltas. Recurrence relation has a connection with several fields (say, difference equations, operator theory), leading to many beautiful asymptotic results.

For historical aspects on the topic of algebraic and analytic aspect of orthogonal Sobolev polynomials we refer the reader to [4, 125, 126, 133].

As in the standard theory, several classes of Sobolev inner products satisfy differential equations. We note that the classical orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel are also Sobolev orthogonal, and they all satisfy differential equations of the form (1.4). The Sobolev orthogonality of the classical polynomials has been discussed in detail in [82, 83, 142]. For example, the Jacobi polynomials $\{P_n^{\alpha,\beta}\}_{n=0}^{\infty}; \alpha, \beta > -1$ are orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_R p(x)q(x)(1-x)^{\alpha}(1+x)^{\beta}H(1-x^2)dx + \int_R p'(x)q'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}H(1-x^2)dx,$$

where H denotes the Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

This orthogonality follows immediately from the well known results that that Jacobi polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}H(1-x^2) \quad \alpha, \beta > -1,$$

and the fact that the first derivative $\{dP_n^{\alpha,\beta}(x)/dx\}_{n=0}^{\infty}$ of the Jacobi polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function $w_{\alpha+1,\beta+1}$. Using the differential properties of some classes of Sobolev inner products we study in Chapter 5 the strong asymptotic behavior of the family of the associated sequence of orthogonal polynomials.

1.2 Strong asymptotic properties of Sobolev orthogonal polynomials

In Chapter 5 we consider the sequence of monic orthogonal polynomials with respect to the Sobolev inner product,

$$\langle f, g\rangle = f(1)\overline{g}(1) + \frac{1}{c} f'(1)\overline{g}'(1) + \int_{-1}^{1} f'\overline{g}'dx, \quad f, g \in \mathbb{P}, \quad c > 0. \tag{1.7}$$

Using the fact that the polynomials of this sequence are eigenfunctions of the fourth order differential operator, cf.[89]

$$\mathcal{L}^{(M)}[u] = (z^2-1)^2u^{(4)} + 4z(z^2-1)u^{(3)} + 2(z-1)((1+2\,c)z + 2\,c + 3)u^{(2)},$$

we give a formula for the strong asymptotic behavior of this sequence.

This section gives a brief overview of the results on strong asymptotic properties of Sobolev polynomials, we refer the reader to the excellent surveys [117, 126, 113] for an exhaustive review of asymptotic properties of Sobolev inner products. We denote through all this work the function $\varphi(z) = z + \sqrt{z^2-1}$, which maps the complement of $[-1,1]$ onto the exterior of the unit circle, where we take the branch of $\sqrt{z^2-1}$ for which $|\varphi(z)| > 1$ whenever $z \in \mathbb{C} \setminus [-1,1]$. This function plays a fundamental role in the asymptotic analysis of some classes of sequences of orthogonal polynomials.

Probably the first asymptotic result was given by Schafke [154] for the derivatives of the Legendre–Sobolev orthogonal polynomials, 10 years after they were introduced by Althammer. Schafke's result reads as

$$Q_n'(x) = nP_{n-1}(x) + \kappa_n^{1/2} O(n^{-3/2}),$$

where Q_n is the n th the monic Legendre-Sobolev polynomials with Sobolev norm $\kappa_n^{1/2}$ and P_{n-1} is the $n-1$ monic Legendre polynomial.

The so called continuous case Sobolev inner product (referred as when the support of the measures in the inner product (1.5) is an infinite set) was first considered in the work of Iserles and others [87], by introducing the concept of coherence of measures

DEFINITION 1.3. *Let (μ_0, μ_1) be a pair of positive Borel measures, and $\{P_n\}_{n=0}^{\infty}$, and $\{T_n\}_{n=0}^{\infty}$, the corresponding sequences of MOP. We say that (μ_0, μ_1) constitutes a 0–coherent (or just coherent) pair, if there exist real non zero constants (coherence parameters) $\sigma_l, \sigma_2, \ldots$, such that*

$$T_n(x) = \frac{P_{n+1}'(x)}{n+1} - \sigma_n \frac{P_n'(x)}{n}, \quad n \geq 1.$$

From [136, 145], Martínez–Finkhlshtein in [130] proved that if (μ_0, μ_1) is a 0–coherent pair of measures, $supp(\mu_0) = [-1,1]$. Then,

$$\lim_{n\to\infty} \frac{Q_n(x)}{T_n(x)} = \frac{2}{\varphi'(x)},$$

uniformly on compact subset of $\mathbb{C} \setminus [-1,1]$, where T_n is the n–th monic orthogonal polynomial with respect to the measure μ_1. This result was extended to the general case of k–coherence in [110, 111]. Meijer in [134] gave the complete classification of all coherent pairs of measures, he proved that necessarily either one of the measures μ_0, μ_1 must be classical, for which the approach of the coherence does not allow to move so far. Perhaps motivated mainly by this drawback Martínez–Finkhlshtein in [127], supported in some ideas in [136, 129], proved that

THEOREM 1.1. *If μ_0, μ_1 are absolutely continuous measures supported on $[-1,1]$ with $\mu_i' = \rho_i, i = 1, 2$, and $\rho_i, i = 1, 2$, satisfy a Szegő condition on $[-1,1]$ then*

$$\lim_{n\to\infty} \frac{Q_n(x)}{T_n(x)} = \frac{2}{\varphi'(x)},$$

locally uniformly in $\mathbb{C} \setminus [-1,1]$

For the special case when the measure μ_1 is the absolutely continuous measure given by the Jacobi weight $\rho_1(x) = (1-x)^{\alpha}(1+x)^{\beta}$ and μ_0 is *admissible* (cf.[136]) the author proved that Theorem 1.1 also holds locally uniformly in $\mathbb{C} \setminus [-1,1]$. Theorem 1.1, (cf. [126, Corollary 5.7]) can be generalized for the case when the measure μ_0 is in the Szegő class and μ_1 is absolutely continuous, with absolutely continuous part ρ_1 satisfying $\frac{1}{\rho_1} \in L^1[-1,1]$. As the author points out, these conditions are not necessary, for instance, they are not necessary for the measure μ_1. Indeed, among the coherent pairs there are measures μ_1 containing mass points outside the support of the absolutely continuous component, and hence not satisfying the Szegő's condition. On the other hand, the assumption on μ_0 is not a necessary condition either. In this sense, in [126] it is posed the problem to find a pair of measures (μ_0, μ_1) with $supp(\mu_0) \subset [-1,1]$ and $supp(\mu_1) = [-1,1]$ such that the asymptotic of Theorem 1.1 is no longer valid. We mention also the extension of the result of Theorem

1.1 to the case of sufficiently smooth Jordan curves or arcs in $\mathbb{C}$ in [127], and for inner products with higher order derivatives, [131].

We finally mention in the continuous case the "balanced" Sobolev inner products which are motivated by the fact that essentially only the second measure μ_1 corresponding to the derivative in the inner product matters and the role of the first one is reduced to "do not disturb". These considerations motivate to "balance" the role of both terms of the inner product

$$\langle f, g\rangle = \int f(x)\, g(x) d\mu_0(x) + \int f'(x)\, g'(x) d\mu_1(x),$$

by considering only monic polynomials. In fact, we can study the asymptotic behavior of polynomials $Q_n(x) = x^n + \dots$ minimizing the norm

$$\|Q_n\|^2 = \int Q_n^2 d\mu_0 + \int \left(\frac{Q_n'}{n}\right)^2 d\mu_1, \quad n \geq 1.$$

The first results in this direction [5] have been obtained assuming coherence of the measures μ_0, μ_1, both supported on $[-1, 1]$.

A different line of research, started about 1988, considered the so-called discrete case, when the measure corresponding to the derivatives, μ_1, in the inner product (1.5) is a finite collection of mass points. Historically, the first results for the strong asymptotic behavior were obtained in the discrete case. Discrete Sobolev orthogonal polynomials appeared in the works of Koekoek, Bavinck and Meijer, who were interested in the Laguerre inner product modified by derivatives evaluated at zero. Since the results were strongly tailored to the specific properties of the Laguerre weight, in 1990 Marcellán and Ronveaux [118] focused on the problem from a more general point of view. Jointly with Alfaro and Rezola then continued this research two years later in [3].

In 1993 Marcellán and Van Assche [120] considered the inner product of the type

$$\langle f, g\rangle_s = \int_{-1}^{1} f(x) g(x) d\mu_0(x) + \lambda f'(c) g'(c),$$

where $c \in \mathbb{R}, \lambda > 0$. Their goal was to compare the Sobolev orthogonal polynomials with the standard orthogonal polynomials associated with the measure μ_0, in order to investigate how the addition of the derivatives in the inner product influences the orthogonal system. With this purpose they assumed that μ_0 is a measure for which the asymptotic behavior of the orthogonal polynomials is known; the most relevant class of this type is the Nevai's class $M(0, 1)$ of orthogonal polynomials with appropriately converging recurrence coefficients. The cornerstone of their approach was the expansion of Q_n, in series of P_n, whose coefficients are asymptotically known. The authors establish that

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \begin{cases} 1 & \text{if} \quad c \in supp(\mu_0), \\ \dfrac{(\phi(z) - \phi(c))^2}{2\phi(z)(z - c)} & \text{if} \quad c \in \mathbb{R} \setminus supp(\mu_0). \end{cases}$$

This shows that the situation is very similar to adding a mass point distribution to the measure μ_0 and comparing the corresponding polynomials. In particular, a zero of Q_n, is attracted by c and the rest accumulate at the support of μ_0.

In 1995 López et al. [102], where using techniques from the analytic theory of Padé approximants extended the above result to the inner product involving a linear differential operator, complex measures and several points in $\mathbb{C}$. In particular, for the inner product

$$\langle f, g\rangle_s = \int_{-1}^{1} f(x) g(x) d\mu_0(x) + \sum_{j=1}^{m} \sum_{i=0}^{N_j} M_{j,i} f^{(i)}(c_j) \mathcal{L}_{j,i}(g; c_j),$$

where $c_j \in \mathbb{C}$ and $\mathcal{L}_{j,i}(g; c_j)$ is the evaluation at $c_j \in \mathbb{C}$ of the linear differential operator $\mathcal{L}_{j,i}$, with constant coefficients acting on g, $M_{j,i} \geq 0$, $N_j > 0$, they studied the asymptotic behavior of the ratios

$$\frac{Q_n^{(\nu)}(x)}{P_n^{(\nu)}(x)}, \quad \nu \in \mathbb{Z}^+, \quad \nu \quad \text{fixed},$$

on compact subsets of $\overline{\mathbb{C}} \setminus supp(\mu_0)$, assuming that the complex measure μ_0 supported on $\mathbb{R}$ belongs to the generalized Nevai's Class $M_{\mathbb{C}}(0,1)$ ([102, Def. 1]). Their result confirms the parallelism between the discrete Sobolev and standard orthogonal polynomials with mass modification of the measure.

A more recent result concerning the discrete case was done in [104] by considering a closed rectifiable Jordan curve in the complex plane Γ, $\{z_1, \ldots, z_m\} \subset \Omega$ is a finite set of points, with Ω denoting the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma$ and $\{\mu_k\}_{k=0}^N$ a set of $N+1$ finite positive Borel measures supported on Γ, where μ_N is such that $d\mu_N(\xi) = \rho_N(\xi)|d\xi|$. The authors obtain the strong asymptotic behavior of the sequence of orthogonal polynomials with respect to the inner product

$$\langle p, q \rangle = \sum_{k=0}^{N} \langle p^{(k)}, q^{(k)} \rangle_k,$$

where

$$\langle p, q \rangle_k = \int_{\Gamma} p(\xi)\overline{q(\xi)} d\mu_k(\xi), \quad k = 0, \ldots, N-1,$$

$$\langle p, q \rangle_k = \int_{\Gamma} p(\xi)\overline{q(\xi)} \rho_N(\xi)|d\xi| + p(Z)\mathcal{A}q(z)^*,$$

$$p(Z) = \left(p(z_1), \ldots, p^{(d_1)}(z_1), p(z_2), \ldots, p^{(d_2)}(z_2), \ldots, p(z_m), \ldots, p^{(d_m)}(z_m)\right),$$

with $\mathcal{A}$ is an Hermitian positive definite matrix of order $M = m + \sum_{i=1}^{m} d_i$.

This inner product generalizes [102] for $N = 0$ and [27], where the authors consider a similar problem. When $N > 0$ and $\mathcal{A} \equiv 0$ (known as the continuous case), the strong asymptotics of Sobolev orthogonal polynomials and their first derivative ($N = 1$) was studied in [127] assuming that μ_0 and μ_1 belong to the Szegő class. A natural extension when $N > 1$ was given in [131].

1.3 Orthogonality with respect to a differential operator

In this section we show the state of the art of the study of orthogonal polynomials with respect to a linear homogeneous differential operator, which is one the objects of study of this thesis.

This relation of orthogonality was introduced in [8] as a generalization of the notion of orthogonal polynomials. There, the authors show that the notion of Tchebyshev system plays a fundamental role in order to solve the problem of the uniqueness of the sequence of the polynomials. A further study of some algebraic and analytic properties of this type of orthogonality is done in [13, 23, 24, 25] for some first and second order linear homogeneous differential operators. Formally, orthogonality with respect to a linear homogeneous differential operator is defined as follows,

DEFINITION 1.4. *Assume that μ is a finite positive Borel measure on the real line and let $\{\rho_k\}_{k=0}^M$ be a set of functions such that,*

$$\int |x^j \rho_k(x)| d\mu(x) < \infty, \quad 0 \leq j < \infty,$$

for all $k = 0, \ldots, M$. Denote

$$\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}, \tag{1.8}$$

an operator acting over the space of polynomials $\mathbb{P}$.

We say that $\{Q_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to the pair $(\mathcal{L}^{(M)}, \mu)$ if $deg[Q_n] \le n$ and

$$\int \mathcal{L}^{(M)}[Q_n](x) P(x) d\mu(x) = 0, \tag{1.9}$$

for any polynomial P such that $deg[P] \le n-1$.

We recall that in the definition of orthogonality with respect to a differential operator given in [8], ρ_M is assumed to be equal to 1, but we shall drop this assumption. The determination of the sequence of these polynomials can be reduced to the solution of a system of n algebraic linear homogeneous equations on the $n+1$ coefficients of Q_n, thus the existence is guaranteed. Unlike systems of orthogonal polynomials, it is not possible to affirm uniqueness up to a constant factor and this turns out to be in general a difficult problem. We say that an index n is normal if for this n the solution is uniquely determined up to a constant factor. For a fixed non negative integer n, Q_n will be referred to as the orthogonal polynomial with respect to the pair $(\mathcal{L}^{(M)}, \mu)$ associated to the index n, which, in general, is not necessarily unique.

Let us see some examples where orthogonality with respect to a differential operator reduces in some sense to orthogonality with respect to an inner product.

1. When $M = 0$, i.e. $\mathcal{L}^{(M)}[f](x) = \rho_0(x) f(x)$, we obtain the classical construction of orthogonal polynomials with respect to a standard inner product

$$\int Q_n(x) P(x) d\mu(x) = 0, \quad deg[P] \le n-1.$$

2. Let $\zeta \in \mathbb{C}$ be fixed and consider the differential operator $\mathcal{L}_\zeta : W^{1,2}(\mu) \to L^2(\mu)$

$$\mathcal{L}_\zeta[f(x)] = f(x) + (x-\zeta) f'(x),$$

where $W^{1,2}(\mu) = \{f \in L^2(\mu) : f' \in L^2(\mu)\}$ is the Sobolev space of index 1. Let us consider a positive Borel measure μ supported on a subset $\Delta \subset \mathbb{R}$. The *polar polynomial* associated to μ, see [12], is defined as the polynomial Q_n of degree n orthogonal with respect to $(\mathcal{L}_\zeta, \mu)$. Let us consider

$$\begin{aligned} \Pi_{0,\zeta} &= 1, \\ \Pi_{n+1,\zeta}(z) &= (z-\zeta) Q_n(z), \quad n \ge 0. \end{aligned}$$

Then it is not difficult to see that the family of polynomials $\{\Pi_{n+1,\zeta}\}_{n=0}^{\infty}$ is orthogonal with respect to the Sobolev inner product

$$\langle f, g \rangle_\zeta = \eta f(\zeta) g(\zeta) + \int_\Delta f'(x)\, g'(x)\, d\mu(x),$$

for some $\eta > 0$. The authors in [13] give a detailed study of this family for the case in which $\mu = \mu_\lambda$, $\lambda > -\frac{1}{2}$, is the (classical) Gegenbauer or ultraspherical measure, i.e. $d\mu_\lambda(x) = (1-x^2)^{\lambda - \frac{1}{2}} dx$.

We also mention that for the case of a first and second order differential operators, the n–th orthogonal polynomial associated to an index n can be interpreted as the the equilibrium points of a flow of a complex potential due to a system of fixed points, cf. [13, 23]

1.3.1 Properties of uniqueness of the sequence of orthogonal polynomials with respect to differential operators in general

As said in the preceding section, the sequence of orthogonal polynomials with respect to differential operators is not necessarily unique. The authors in [8] find that the notion of T–system results to be a sufficient condition for the normality of the sequence for linear homogeneous differential operators in general. We remind that,

DEFINITION 1.5. *A set $\{u\}_{k=0}^{n}$ of continuous functions on Δ is called a Tchebychev system (T–system) on Δ if any linear combination*

$$\sum_{k=0}^{n} \alpha_k u_k,$$

has at most n zeros on this interval. If for each $0 \leq n' \leq n$; the set of functions $\{u_k\}_{k=0}^{n'}$ forms a T-system it is called a Markov system (M–system).

A sufficient condition for uniqueness is that any polynomial satisfying (1.9) has exact degree n. In fact, because of the linearity in the construction, the difference of two solutions is also a solution; therefore, two different solutions of equal degree not multiples of each other generate another one of smaller degree contradicting our assumption. Based mainly on this fact, the sufficient conditions for the question of the uniqueness of the sequence of orthogonal polynomials with respect to linear homogeneous differential operators in general are given in the next three theorems, cf. [8].

THEOREM 1.2. *Given $\mathcal{L}^{(M)}$ as in (1.8), let us assume that $\left\{\mathcal{L}^{(M)}[x^{\nu}]\right\}_{\nu=0}^{n}$ is an M-system on $supp(\mu)$. Then $deg[Q_n] = n$*

This result establishes a correspondence between T–systems and fundamental systems of solutions of linear differential equations because any fundamental solution $(u_0, \ldots, u_{M-1})$ of $\mathcal{L}^{(M)}[u] = 0$ satisfies $W(u_0, \ldots, u_{M-1})$. Therefore, any such solution is a T-system.

THEOREM 1.3. *Let $\{u_0, \ldots, u_{M-1}\}$ be a fundamental system of solutions of $\mathcal{L}^{(M)}[u] = 0$. Let's assume that $n \in \mathbb{N}$ is given and that*

$$\{(u_0^{(\nu)}, \ldots, u_{M-1}^{(\nu)})\},$$

is a T-system for $\nu = 1, 2, \ldots, n+1$. Then $deg[Q_n] = n$ where Q_n is the n-th orthogonal polynomial with respect to $(\mathcal{L}^{(M)}, \mu)$.

The preceding theorem gives a condition of normality in terms of a fundamental system of solutions. This condition in terms of the coefficients of the differential operator gives,

THEOREM 1.4. *Assume that $\mathcal{L}^{(M)}$ has infinitely differentiable coefficients $\{\rho_k\}_{k=0}^{M}$ on $supp(\mu)$. Define recurrently the system of functions $\{\rho_{k,n'}\}_{k=0}^{M}$, $n^{'} = 1, 2, \ldots,$ as follows*

$$\{\rho_{k,0} := \rho_k\}_{k=0}^{M},$$

is a T-system for $\nu = 1, 2, \ldots, n+1$. Then $deg[Q_n] = n$ if for all $n^{'} = 1, 2, \ldots, n$ we have $\rho_{0,n'}(x) \neq 0$, $x \in supp(\mu)$.

Using the above results, the authors [8] prove, for some cases of differential operators, the normality of the associated sequence of orthogonal polynomials.

1.3.2 The polar polynomials

A more detailed analysis of the sequence of orthogonal polynomials with respect to some classes of first order operator was started in [12, 13] by introducing the *polar polynomials* which were already defined in 2) of Section 1.3 as the polynomials orthogonal with respect to the pair $(\mathcal{L}_\zeta, \mu)$ where $\mathcal{L}_\zeta$ is the operator

$$\mathcal{L}_\zeta = \frac{d^0}{dx^0} + (x-\zeta)\frac{d}{dx},$$

and μ is a positive Borel measure with support contained in $\mathbb{R}$. When $d\mu_\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}dx, \lambda > -\frac{1}{2}$, that is, the Gegenbauer measure, the authors define the sequence $\{Q_n\}_{n=0}^\infty$ of *Gegenbauer polar polynomial* as the sequence of orthogonal polynomials with respect to the pair $(\mathcal{L}_\zeta, \mu_\lambda)$. In this case, it is not difficult to see that the sequence of monic polynomials is uniquely determined. This type of orthogonality is then applied to the study of the family of Sobolev–Gegenbauer polynomials $\{\Pi_{n,\zeta}\}_{n=0}^\infty$ with pole $\zeta \in \mathbb{C}$ which are defined as the sequence of orthogonal polynomials with respect to the discrete–continuous inner product,

$$\langle f, g\rangle_\zeta = \eta f(\zeta)g(\zeta) + \int_{-1}^{1} f'(x)\, g'(x)\, d\mu_\lambda(x). \tag{1.10}$$

Then, the following relation holds,

$$\Pi_{n+1,\zeta}(z) \quad = \quad (z-\zeta)Q_n(z), \quad n \geq 0.$$

The inner product (1.10) is a subclass of the more general inner product

$$\langle f, g\rangle_\zeta = \int f(x)\overline{g}(x)d\mu_0(x) + \int f'(x)\,\overline{g}'(x)\,d\mu_1(x),$$

where $\Delta_0, \Delta_1 \subset \mathbb{C}$. This class of inner product was introduced in [35] in order to study the behavior of the best polynomial approximation of absolutely continuous functions in the norm associated with this inner product. In this sense [67] is a continuation of the works with higher derivatives.

Using the approach of the orthogonality with respect to differential operator, the authors [13] prove a series of algebraic and analytical results. The family of Sobolev–Gegenbauer polynomials $\{\Pi_{n,\zeta}\}_{n=0}^\infty$, can be generated from the recursion relation,

THEOREM 1.5. *The sequence of Gegenbauer polar polynomials with pole $\zeta \in \mathbb{C}$, satisfies the following recurrence relation,*

$$\Pi_{n+2,\zeta}(z) = z\Pi_{n+1,\zeta}(z) + a_n\Pi_{n,\zeta}(z) + b_n\Pi_{1,\zeta}(z),$$

for $n > 1$, where $\Pi_{0,\zeta}(z) = 1$,

$$a_n = \frac{4(n^2 + 2\lambda(n+1)}{(n+\lambda)(n+\lambda-1)} \quad \textit{and} \quad b_n = -C_{n+1}^\lambda(\zeta).$$

Here C_n^λ denotes the monic Gegenbauer or ultraspherical polynomial orthogonal with respect to the measure $d\mu_\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}dx$.

The next theorem gives us the zero distribution, for fixed n, of the nth discrete–continuous Sobolev–Gegenbauer polynomials and its multiplicity.

THEOREM 1.6. *The discrete–continuous Sobolev–Gegenbauer polynomials $\{\Pi_{n,\zeta}\}_{n=0}^\infty$ with pole $\zeta \in \mathbb{C}$, satisfy*

1. *If n is even and $\zeta \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, then $-\zeta$ is a zero of $\Pi_{n,\zeta}$.*

2. *The zeros of the polynomial $\Pi_{n,\zeta}$ have multiplicity at most 2 and their multiple zeros are located on $[-1,1]$.*

3. *All the zeros of $\Pi_{n,\zeta}$ are located on the lemniscate*

$$\mathbf{E}_n(\zeta) := \left\{ z \in \mathbb{C} : \prod_{k=1}^{n} |z - x_{n,k}| = \rho_n(\zeta) \right\},$$

where $\rho_n(\zeta) = \prod_{k=1}^{n} |\zeta - x_{n,k}|$ and $\{x_{n,1}, x_{n,2}, \cdots, x_{n,n}\}$ are the n zeros of the polynomial $\widehat{C}_n^{\lambda}$.

The uniform boundedness of the set of zeros of the discrete–continuous Sobolev–Gegenbauer polynomials $\{\Pi_{n,\zeta}\}$, which is the set of source points, can be seen in the next result

THEOREM 1.7. *Given $\zeta \in \mathbb{C}$, define $\Delta_\zeta = \sup_{x\in[-1,1]} |\zeta - x|$ and $\delta_\zeta = \inf_{x\in[-1,1]} |\zeta - x|$, then*

1. *All zeros of the nth discrete–continuous Sobolev–Gegenbauer polynomials $\{\Pi_{n,\zeta}\}_{n=0}^{\infty}$ are contained in $|z| \leq 1 + \Delta_\zeta$.*

2. *If $\delta_\zeta > 1$, the zeros of the discrete–continuous Sobolev–Gegenbauer polynomial $\{\Pi_{n,\zeta}\}$, with $\zeta \in \mathbb{C}$ are simple and for n sufficiently large, contained in the exterior of the ellipse $|z+1| + |z-1| = 2a$, where $1 < a < \delta_\zeta$.*

Finally, for the behavior of the zeros of the discrete–continuous Sobolev–Gegenbauer polynomial, with n sufficiently large we have

THEOREM 1.8. *Assume $\delta_\zeta > 1$, and let $\Pi_{n,\zeta}$ be the n-th discrete–continuous Sobolev–Gegenbauer monic polynomial with $\zeta \in \mathbb{C} \setminus [-1,1]$. Then the accumulation points of zeros of $\{\Pi_{n,\zeta}\}$ are located on the ellipse*

$$\mathbf{E}(\zeta) := \left\{ z = x + iy \in \mathbb{C} : \frac{x^2}{(\rho^2(\zeta)+1)^2} + \frac{y^2}{(\rho^2(\zeta)-1)^2} = \frac{1}{4\rho^2(\zeta)} \right\},$$

where $\rho(\zeta) := |\zeta + \sqrt{\zeta^2 - 1}|$ and the branch of the square root is chosen so that $|\zeta + \sqrt{\zeta^2-1}| > 1$.

In Chapter 4, Section 4.6 we consider the class of finite positive measures supported on $[-1,1]$ defined as $d\mu(x) = \dfrac{d\mu_T(x)}{\rho(x)}$ with $\rho(z) = r\prod_{i=1}^{m}(z - \nu_i)$ a nonnegative polynomial on $[-1,1]$ and $d\mu_T(x) = \dfrac{1}{\sqrt{1-x^2}}dx$ is the first kind Tchebychev measure and we obtain a curve which contains the accumulation points of the zeros of these polynomials and a formula for the strong asymptotic behavior of these polynomials on $\mathbb{C} \setminus [-1,1]$.

1.4 On matrix orthogonal polynomials systems

Chapter 6 of this thesis deals with matrix orthogonal polynomials and more specifically, with differential properties that some of classes of these systems of polynomials satisfy. In this section we introduce some basic facts and definitions concerning to matrix orthogonality on the real line.

Matrix valued orthogonal polynomials were introduced by M.G. Krein [95, 97], more than fifty years ago. It is not clear if Krein had in mind some applications of these polynomials or his main motivation was a functional one in terms of extensions of symmetric operators. After the work of Krein many other authors have contributed to the theory of matrix valued orthogonal polynomials on the real line, see for instance [9, 11, 14, 30, 39, 41, 42, 43, 44, 46, 68, 88, 116, 114, 119], and their references (this list is not exhaustive).

Let $\mathcal{M}_N$ denote the ring of all $N \times N$ complex valued matrices; we denote by A^* the Hermitian conjugate of $A \in \mathcal{M}_N$. Consider the set $\mathcal{P}$ of all polynomials $\mathcal{P}$ in $z \in \mathbb{C}$ with coefficients in $\mathcal{M}_N$. The set $\mathcal{P}$ can be considered either as a right module or a left module over $\mathcal{M}_N$ according to the case in which we have a multiplication operation defined by the right or the left respectively; clearly the conjugation makes the left and right structures isomorphic. A matrix polynomial P_n denotes an element in $\mathcal{P}$ of degree at most n, that is, a combination $P(t) = A_0 + A_1 t + \cdots + A_n t^n$, where $A_0, \ldots, A_n \in \mathcal{M}_N$. A weight matrix is defined as

DEFINITION 1.6. *We say that a $N \times N$ matrix of measures supported on the real line W is a weight matrix if*

1) *$W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$,*

2) *W has finite moments,*

3) *$\int P(t)dW(t)P^*(t)$ is non singular if the leading coefficient of the matrix polynomial P is nonsingular.*

Given a weight matrix W, we can consider the left inner product defined by the skew symmetric bilinear form on the space $\mathcal{P}$ as a left module,

$$\langle P, Q \rangle_L = \int P(t)dW(t)Q^*(t). \tag{1.11}$$

Then, a sequence of matrix polynomials $\{P_n\}_{n=0}^{\infty}$, $deg[P_n] = n$, is orthogonal with respect to (1.11) if

$$\int P_n(t)dW(t)P_m^*(t) = \Delta_n \delta_{n,m},$$

where $\Delta_n, n \geq 0$, is a positive definite matrix. When $\Delta_n = I$, where I is the $N \times N$ identity matrix, we say that the matrix polynomials $\{P_n\}_{n=0}^{\infty}$ are orthonormal.

By using a standard Gram–Schmidt process it is not difficult to see that condition 3) of Definition 1.6 is necessary and sufficient to guarantee the existence of a sequence of matrix polynomials orthonormal with respect to (1.11). This condition is fulfilled, in particular, when W is positive definite at infinitely many points of the support of W.

We remark that for the case in which we have the space $\mathcal{P}$ as a right module then it would be appropriate to consider the right inner product defined by the skew bilinear form

$$\langle P, Q \rangle_R = \int P^*(t)dW(t)Q(t), \tag{1.12}$$

indeed, if $\{P_n\}_{n=0}^{\infty}$ is a sequence of orthonormal matrix polynomials with respect to (1.12) then it would be desirable that $\{P_n \Gamma_n\}_{n=0}^{\infty}$, where $\{\Gamma_n\}_{n=0}^{\infty}$ is a sequence of unitary matrices, results also a sequence of orthonormal matrix polynomials and this is possible only if we define the bilinear form as (1.12).

As remarked before, $\{P_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to (1.11) if and only if $\{P_n^*\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to (1.12). Since the multiplication by the left is more natural, it is usual in the theory of matrix orthogonal polynomials to consider left inner products. We say then: the sequence of orthogonal polynomials with respect to W when referring to the sequence of orthogonal polynomials with respect to (1.11).

One might be tempted to think of $\langle P, P \rangle_L$ (or $\langle P, P \rangle_R$) as some kind of norm, but that is doubtful. Even if W is supported at a single point, x_0, with $W(t) = N^{-1} I$, this "norm" is essentially the absolute value of $A = P(x_0)$ which is known not to obey the triangle inequality (see [159, Sect. I.1]). However, if one looks at

$$\|P\|_L = (Tr[\langle P, P \rangle_L])^{1/2},$$

where $Tr[.]$ denotes the trace of a matrix, then one have a norm, see [38].

Just as in the scalar case, a sequence of monic orthogonal matrix polynomials $\{P_n\}_{n=0}^{\infty}$ satisfies a three term recurrence relation

$$A_n P_{n-1}(t) + B_n P_n(t) + P_{n+1}(t) = tP_n(t),$$

with the initial condition $P_{-1} = 0$, the null matrix, and P_0 the identity matrix, and where the B_n's are Hermitian and the A_n's are non singular, see [6, 42, 55].

Some very important results of the theory of scalar valued orthogonal polynomials, like Favard's Theorem and Markov's Theorem have been extended to the matrix valued case, see [42, 43, 44, 58, 55], and many more still need to be investigated in the new context of the matrix valued orthogonal polynomials.

When dealing with weight matrices it is convenient to consider the following equivalence relation: We say that two weight matrices W_1, W_2 are similar if there exists a nonsingular matrix T (independent of t) such that $W_1 = TW_2T^*$.

Given this notion of similarity, it is important to single out two special cases. We say that a weight matrix W reduces to a lower size if there exists a nonsingular matrix T for which

$$W(t) = T \begin{pmatrix} Z_1(t) & 0 \\ 0 & Z_2(t) \end{pmatrix} T^*,$$

where Z_1 and Z_2 are weight matrices of lower size. Notice that the orthonormal matrix polynomials with respect to a W are then

$$P_n(t) = T \begin{pmatrix} P_{n,1}(t) & 0 \\ 0 & P_{n,2}(t) \end{pmatrix} T^*, \quad n \geq 0,$$

where $\{P_{n,i}\}_{n=0}^{\infty}$ are the orthonormal matrix polynomials with respect to $Z_i, i = 1, 2$. Analogously, we say that W reduces to scalar weights if there exists a nonsingular matrix T for which

$$W(t) = TD(t)T^*,$$

with D diagonal. This is clearly an extreme case of the situation considered earlier. According to our equivalence relation, to say that W does not reduce to lower size is just to say that there is no block diagonal weight matrix in the equivalence class of W, while weight matrices reducible to scalar weights are, precisely, those corresponding to the class of diagonal weights. Diagonal weights, as a collection of N scalar weights, belong to the study of scalar orthogonality more than to the matrix one. Unfortunately, this is the case of many examples of orthogonal matrix polynomials which can be found in the literature. We observe, however, that in [78] one finds a notion of similarity for the pair consisting of the weight and the differential operator. This notion allows one to distinguish among certain situations that are considered equivalent under the present definition. See [78, Example 5.1].

If we assume that for some real number a, $W(a) = I$, then W reduces to scalar weights if and only if $W(t)W(s) = W(s)W(t)$ for all t, s. This commutativity condition on the weight matrix W gives a convenient way of checking if one is dealing with a case that reduces to scalar weights.

The systems of orthogonal polynomials associated with the names of Hermite, Laguerre, Bessel and Jacobi (including the special cases named after Tchebychev, Legendre, and Gegenbauer) are the most extensively studied and widely applied systems. These four families of orthogonal polynomials are called collectively the classical orthogonal polynomials and were already introduced in Section 1.1. These systems of polynomials share the fact that they are the unique families of orthogonal polynomials with respect to a positive positive measure satisfying a second order linear differential equation with polynomial coefficients, [19].

For the case of matrix orthogonal polynomials, more than 50 years have been necessary to produce the first examples of orthogonal matrix polynomials $\{P_n\}_{n=0}^{\infty}$ satisfying second-order linear differential equations of the form

$$P_n^{''}(t)F_2(t) + P_n^{''}(t)F_1(t) + P_n^{''}(t)F_0 = \Gamma_n P_n(t), \quad n \geq 0, \tag{1.13}$$

where F_2, F_1, F_0 are matrix polynomials (which do not depend on n) of degrees not larger than $2, 1, 0$, respectively, and Γ_n are matrices (see [49, 76, 78]). These families of orthogonal matrix polynomials are among those that are likely to play in the case of matrix orthogonality the role of the classical families of Hermite, Laguerre and Jacobi in the case of scalar orthogonality.

Equation (1.13) for the polynomial P_n is equivalent to say that P_n is an eigenvector of the right–hand side second order differential operator

$$l_{2,R} = D^2 F_2(t) + D^1 F_1(t) + D^0 F_0, \tag{1.14}$$

with eigenvalue Γ_n. When the sequence $\{P_n\}_{n=0}^{\infty}$ is orthonormal and the eigenvalues $\Gamma_n, n \geq 0$ are Hermitian, the differential equation (1.13) is equivalent to the symmetry of the operator (1.14) with respect to W (see [45, Lemma 4]). If Γ_n's are not Hermitian, then the operator $l_{2,R}$ can be decomposed as $l_{2,R} = l_{2,R,1} + \imath\, l_{2,R,2}$, where $l_{2,R,1}$ and $l_{2,R,2}$ are second order differential operators of the form (1.14) symmetric with respect to W (see [81]). We recall that the symmetry of a differential operator l with respect to a weight matrix W is defined in the usual way:

$$\int l[P] dW Q^* = \int P dW (l[P])^*,$$

for any matrix polynomials P and Q.

In Chapter 6 we introduce a new family of matrix orthogonal polynomials satisfying a second order differential equation giving also their recurrence relations and Rodrigues' formulas. In the next sections of this introductory chapter we give a brief overview of the known families of matrix polynomials satisfying second order differential equations like (1.13).

1.4.1 Left inner products and right–hand side second order differential operators

It is a natural question to ask why the coefficients in (1.13) appear on the right side of the argument. In considering differential operators it is customary to write them as linear combinations of products of functions of t multiplied on the right by powers of the differentiation operator. When we deal with the matrix valued case where nothing is assumed to commute it is clear that, using the notation of (1.13) and (1.14) the adjoint of a term like

$$F(t) D^2 P(t),$$

is given by

$$P^*(t) D^2 F(t),$$

i.e. we go from a left-hand differential operator acting on P to a right-hand operator acting on P^*. We could therefore be considering right–hand side operators

$$l_{2,R} = D^2 F_2(t) + D^1 F_1(t) + D^0 F_0,$$

as well as left–hand side operators

$$l_{2,L} = F_2(t) D^2 + F_1(t) D^1 + F_0 D^0.$$

Right-hand side operators are more natural and interesting in relation with matrix inner products defined by a weight matrix W in the usual form by (1.11), while left–hand side differential operators are more convenient when the inner product is defined in the *less* natural way defined by (1.12).

The reason is the following: when inner products of the form (1.11) are considered, the natural way to expand a matrix polynomial in terms of the sequence of orthonormal polynomials $\{P_n\}_{n=0}^{\infty}$ is to put $P(t) =$

$\sum_n \Lambda_n P_n(t)$, that is, placing the matrix coefficients on the left; otherwise the coefficients Λ_n interfere with the orthogonality of $\{P_n\}_{n=0}^{\infty}$ since in (2.1) the polynomial P multiplies the weight W on the left. Analogously, for (1.12), the natural expansion takes the form $P(t) = \sum_n P_n(t)\Lambda_n$. It turns out that righthand–side operators are left linear but not right linear: that is, $l_{2,R}[CP] = Cl_{2,R}[P]$, P a matrix function and C a constant matrix, but, in general, $l_{2,R}[PC] \neq l_{2,R}[P]C$; analogously left–hand side operators are right linear but not left linear: $l_{2,L}[PC] = l_{2,L}[P]C$, but, in general, $l_{2,L}[CP] \neq Cl_{2,L}[P]$. This lack of left linearity has the following undesirable consequence, [45, Lemma 2.1]:

LEMMA 1.1. *Let W be a positive definite matrix of measures and $\{P_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials with respect to it. Then, for a right-hand side second order differential operator $l_{2,R}$ the following conditions are equivalent:*

a) The operator $l_{2,R}$ is symmetric with respect to the inner product of the form (1.11)*, i.e.*

$$\langle l_{2,R}[P], Q\rangle_L = \langle P, l_{2,R}[Q]\rangle_L,$$

for any matrix polynomials P and Q.

b) The orthonormal polynomial P_n is a eigenvector of $l_{2,R}$ with a Hermitian left eigenvalue $\Gamma_n : l_{2,R}[P_n] = \Gamma_n P_n, n = 0, 1, \ldots$

For a left-hand side operator a) also implies b) but, in general, b) does not imply a).

There is another reason to consider left-hand side operators as less interesting (than right hand ones) when the inner product (1.11) is used: as proved in [45, Th. 3.2], in the matrix case all the examples of weight matrices having a symmetric left-hand side second order operator reduce to the scalar classical examples.

For all the reasons given above, in the sequel, we shall consider sequences of orthogonal polynomials with respect to a left inner product defined by (1.11) and right hand side operators $l_{2,R}$. For brevity, we shall refer to the sequence of matrix orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ with respect to a left inner product (1.11) as the sequence of matrix orthogonal polynomials with respect to the weight W.

1.4.2 The matrix Pearson equations for second order matrix differential operators

In Section 1.4 we mentioned that the existence of a sequence $\{P_n\}_{n=0}^{\infty}$ of matrix orthonormal polynomials and eigenvalues $\Gamma_n, n \geq 0$ for the differential equation (1.13) is equivalent to the symmetry of the operator (1.14) with respect to W. In this section we show how to convert the condition of symmetry for the pair made up of a weight matrix W and a right–hand side second order differential operator $l_{2,R}$, which results to be the the matrix analogous to the Pearson equation $(f_2 w)' = f_1 w$ of the scalar case. To simplify, in the sequel we shall consider weight matrices $dW = W(t)dt$ that have a smooth density W with respect to the Lebesgue measure. The case of study of weight matrices W which are not absolutely continuous with respect to the Lebesgue measure remains open.

In the last few years two different methods have been developed for finding weight matrices W having symmetric second-order differential operators like (1.14). One of these methods has been developed by F.A. Grünbaum, I. Pacharoni and J.A. Tirao, and consists, basically, in to find examples of orthogonal matrix polynomials satisfying (1.13) by manipulating certain matrix spherical functions which appear in the representation of certain groups (for instance, the complex projective plane realized as the symmetric space G/K with $G = SU(3)$ and $K = U(2)$); see [77, 79, 80, 144] or [143].

The other method, developed by F.A. Grünbaum and A. Durán (see [49, 50]), basically consists in solving certain matrix differential equations that imply the symmetry of second-order differential operators like (1.14) with respect to a weight matrix, that is

THEOREM 1.9. *For a weight matrix $dW = W(t)dt$, $t \in (a,b)$ (a and b finite or infinite) and matrix polynomials F_2, F_1, F_0 of degrees not larger than $2, 1$ and 0, the symmetry of the second order differential operator* (1.14) *with respect to W follows from the set of equations*

$$F_2 W = W F_2^*, \tag{1.15}$$

$$2(F_2W)' - F_1 W = W F_1^*, \quad (F_2W)'' - (F_1W)' + F_0 W = W F_0^*, \tag{1.16}$$

under the boundary conditions

$$\lim_{t\to a^+, b^-} t^n F_2(t)W(t) = 0, \qquad \lim_{t\to a^+, b^-} t^n[(F_2(t)W(t))' - F_1(t)W(t)] = 0, \quad n \geq 0. \tag{1.17}$$

Equations (1.15),(1.16) are the analogous to the Pearson equation for the scalar case (see [49] or [80]).

We can sort out the class of weight matrices having symmetric second-order differential operators in two disjoint subclasses. On the one hand are those weight matrices having at least one of these operators with diagonal leading coefficient. It is known a rich family of inhabitants living in this subclass, and all of them can be characterized as

$$e^{-t^2} TT^*, \quad t^\alpha e^{-t} TT^*, \quad \text{or} \quad (1-t)^\alpha (1+t)^\beta TT^*,$$

where T is a matrix function satisfying, respectively, a differential equation like

$$T^{'} = (A + B\,t)T, \quad T^{'} = (A + t/B)T, \quad \text{or} \quad T^{'} = \left(\frac{A}{1-t} + \frac{B}{1+t}\right)T, \tag{1.18}$$

for certain matrices A and B. These examples always have associated a symmetric second-order differential operator with diagonal leading coefficient F_2 equal to I, tI or $(1-t^2)I$, respectively. In the first method this leading coefficient F_2 is equal to $(1-t^2)I$, and the second method provides examples with F_2 equal to I, tI or $(1-t^2)I$. Some of these weight matrices, but not all, also have other second-order symmetric differential operators with non-diagonal leading coefficient. For the classes of matrix orthogonal polynomials satisfying equations (1.18), the reader can check [49, 50, 52]. The existence of weight matrices having several linearly independent second-order symmetric differential operators is a new phenomenon. In the scalar case, the classical families of Hermite, Laguerre and Jacobi have, up to a multiplicative constant, only one symmetric second-order differential operator (that is also the case with the Bessel polynomials). The existence of such examples of weight matrices follows as a consequence of [77], which already contains two second-order differential operators (one of them with non-diagonal leading coefficient) acting on matrix spherical functions related to $SU(3)$, and [79, Sect. 5], where how to connect the matrix spherical functions of [77] and orthogonal matrix polynomials is explained.

On the other hand, we can consider the subclass of those weight matrices for which all their symmetric second-order differential operators have non-diagonal leading coefficient. A step in the exploration of this second class was given in [47], and is supported in the following

THEOREM 1.10. *Let Ω be an open set of the real line. Let F_2, F_1, F and T be twice differentiable $N \times N$ matrix functions on Ω, (with $T(t_0)$ nonsingular for certain $t_0 \in \Omega$), and define $W(t) = T(t)T^*(t)$. Under the assumptions*

$$F_2 W = W F_2^*, \tag{1.19}$$

$$T'(t) = F(t)T(t), \tag{1.20}$$

$$F_1 = F_2 F + F F_2 + F_2', \tag{1.21}$$

we have

1. *The weight matrix W satisfies the first order differential equation*

$$2(F_2W)' = F_1W + WF_1^*.$$

2. *For a given matrix F_0, the second order differential equation*

$$(F_2W)'' - (F_1W)' + F_0W = WF_0^*,$$

holds if and only if the matrix function

$$\chi = T^{-1}(-FF_2F - F'F_2 - FF_2' + F_0)T, \tag{1.22}$$

is Hermitian at each point of Ω.

Using Theorem 1.10, in [47] a class in which Hermite and Laguerre weights appear together in the weight matrix is given. The class is a weight matrix of size $N \times N$ depending on N parameters having a symmetric second–order differential operator like (1.14) with differential coefficient

$$F_2(t) = \mathcal{E}_{1,1} + 2t\sum_{i=2}^{N}\mathcal{E}_{i,i} + t(2t-1)\sum_{i=2}^{N}a_{1,i}\mathcal{E}_{1,i},$$

where $\mathcal{E}_{i,j}$ stands for the matrix with entry (i,j) equal to 1, and all other entries equal to zero. The weight matrix W is given in the factorized form TT^*, with $T = PT_0$, where P is a certain matrix polynomial and T_0 is the diagonal matrix

$$T_0(t) = e^{-t^2/2}\mathcal{E}_{1,1} + e^{-t^2/2}t_+^{\alpha}\sum_{i=2}^{N}t^{N-i}\mathcal{E}_{i,i},$$

where

$$t_+^{\alpha} = \begin{cases} t^{\alpha}, & t > 0, \\ 0 & t \leq 0. \end{cases}$$

The matrix function T satisfies a first–order differential equation $T^{'} = FT$, where $F = A/t + B + Ct + Dt^2$, for certain matrices A, B, C and D. This differential equation for T should be compared with those in (1.18).

Using Theorem 1.10, we find in Chapter 6 a new class of weight matrices for which all their symmetric second-order differential operators have non-diagonal leading coefficient.

1.4.3 Rodrigues' formulas for orthogonal matrix polynomials satisfying second order differential equations

It is well known that the classical families of Hermite, Laguerre and Jacobi satisfies a second order differential equation of the form

$$f_2P_n^{''} + f_1P_n^{'} = \lambda_nP_n, \quad n \geq 0,$$

where $f_i, i = 1, 2, 0$ are polynomials of degree not larger than i and independent of n. They can be characterized also by using the Rodrigues formula:

$$P_n = (f_2^n w)^{(n)}/w,$$

where w is the corresponding weight and $f_2 = 1, t$, and $1 - t^2$, respectively. Each one of these characterization is the result of a different effort, and they are usually associated to names of S.Bochner and E.Hildebrandt. Actually these properties follow from the Pearson equation for the weight function w: $(f_2 w)' = f_1 w$, which also characterizes the three classical weights of Hermite, Laguerre and Jacobi, see [34, 2].

For the matrix case we have a different situation. Orthogonal matrix polynomials satisfying a differential equation like (1.13) with $F_2 = f_2 I$ and f_2 is a scalar polynomial of degree not larger than 2, do not satisfy, in general, a Rodrigues' formula of the form

$$P_n = C_n \, (f_2^n W)^{(n)} \, W^{-1}, \quad n \geq 0, \tag{1.23}$$

where $C_n, n \geq 0$ are non singular matrices. Indeed, by setting $n = 1$, that Rodrigues' formula gives the equation

$$(f_2 W)' = \Psi \, W, \tag{1.24}$$

where Ψ is a matrix polynomial of degree 1 (which is the first orthogonal polynomial with respect to W). But weight matrices satisfying (1.15),(1.16) do not satisfy in general, an equation like (1.24). It was conjectured by Durán and Grünbaum in [51] that weight matrices satisfying an equation like (1.24) reduce to scalar weights. The conjecture has turned out to be correct and has been proved independently by Cantero, Moral and Velázquez in [31]. All these facts suggest that scalar type Rodrigues formulas of the form (1.23) seem to be not useful to define orthogonal polynomials.

Instead of fromula (1.23), it seems reasonable to look for some modified Rodrigues' formula. The first instance of that modified Rodrigues' formula appears in [49]: the expression

$$P_n(t) = \left[e^{-t^2} \begin{pmatrix} 1 + |a|^2 t^2 + |a|^2 \frac{n}{2} & a\,t \\ \overline{a}\,t & 1 \end{pmatrix} \right]^{(n)} W^{-1}(t),$$

defines a sequence of matrix orthogonal polynomials with respect to the weight matrix

$$e^{-t^2} \begin{pmatrix} 1 + |a|^2 t^2 & a\,t \\ \overline{a}\,t & 1 \end{pmatrix}.$$

Afterwards, Rodrigues' formulas of the form

$$P_n(t) = (f_2^n \rho \xi_n)^{(n)} W^{-1}, \tag{1.25}$$

where $W = \rho Z, \xi_n$ are certain matrix functions, $\rho(t) = e^{-t^2}, t^\alpha$ or $(1-t)^\alpha (1+t)^\beta$ and f_2 is equal to $1, t$ and $1 - t^2$ respectively have been found for other families of orthogonal polynomials of size 2×2, see [53, 54, 57, 56].

In all these examples, the functions ξ_n are simple enough as to make the Rodrigues formulas (1.25) useful for an explicit calculation of the sequence P_n of orthogonal polynomials with respect to W, when the matrix function W is a matrix polynomial with bounded degree not depending on n. That is the case of the Rodrigues formulas given in [53, 57, 56]. The situation is more involved when the function Z is not a matrix polynomial: one can still hope to find a Rodrigues formula, but in general the functions ξ_n are not going to be polynomials, as the case of Rodrigues formulas in [54].

Up to date, we have the following result, given in [48], for the construction of Rodrigues' formulas of the form (1.25),

THEOREM 1.11. *Let F_2, F_1, and F_0 be matrix polynomials of degrees not larger than* 2, 1, *and* 0, *respectively. Let W, R_n be $N \times N$ matrix functions twice and n times differentiable, respectively, in an open set Ω of the real line. Assume that W is nonsingular for $t \in \Omega$ and that satisfies the identity (1.15), and the differential equations* (1.16). *Define the functions $\mathtt{P}_n$, $n \geq 1$, by*

$$\mathtt{P}_n = R_n^{(n)} W^{-1}. \tag{1.26}$$

If for a matrix Λ_n, the function R_n satisfies

$$(R_n F_2^*)'' - \big(R_n\big[F_1^* + n(F_2^*)'\big]\big)' + R_n\left[F_0^* + n(F_1^*)' + \binom{n}{2}(F_2^*)''\right] = \Lambda_n R_n, \tag{1.27}$$

then the function $\mathtt{P}_n$ satisfies

$$\mathtt{P}_n''(t)F_2(t) + \mathtt{P}_n'(t)F_1(t) + \mathtt{P}_n(t)F_0(t) = \Lambda_n \mathtt{P}_n(t). \tag{1.28}$$

By using the above theorem, in [47] it is given a Rodrigues of the form (1.25) for the weight matrices of arbitrary size $N \times N$ for the following weight matrices

$$W_1(t) = e^{-t^2} e^{At} e^{A^* t},$$

where A is the $N \times N$ nilpotent matrix

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \tag{1.29}$$

and $v_i, I = 1, \dots, N-1$, are complex numbers satisfying

$$(N-i-1)|v_i|^2 |v_{N-1}|^2 - 2i(N-i)|v_{N-1}|^2 + 2(N-1)|v_i|^2 = 0.$$

The matrix $\mathcal{E}_{i,j}$ stands for the matrix with entry (i,j) equal to 1 and 0 otherwise.

For the weight matrix,

$$W_2(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}J} e^{-t} t^{\frac{1}{2}J} e^{A^* t}, \quad \alpha > -1, \quad t \in (0, +\infty)$$

where A as in (1.29) and J is the diagonal matrix

$$J = \sum_{i=1}^{N} (n-i)\mathcal{E}_{i,i}, \tag{1.30}$$

and the complex numbers $v_i, I = 1, \dots, N-1$ satisfying

$$(N-i-1)|v_i|^2 |v_{N-1}|^2 - i(N-i)|v_{N-1}|^2 + (N-1)|v_i|^2 = 0.$$

And for the weight matrix

$$W_3(t) = t^\alpha (1-t)^\beta (2(1-t))^C (2t)^{\frac{1}{2}J} (2t)^{\frac{1}{2}J^*} (2(1-t))^{C^*}, \quad \alpha, \beta > -1, \quad t \in (0,1)$$

where J as in (1.30),

$$C = (N-1)I - J + A,$$

and A is the matrix defined by (1.29) for

$$v_i = -\sqrt{\frac{2i(N-i)(\beta+i-k)}{k+N-1-i}}, \quad 0 < k < \beta + 1, \quad i = 1, \dots, N-1.$$

In Chapter 6 we find a Rodrigues' formula for size 2×2 of a new class of matrix orthogonal polynomials of size $N \times N$ satisfying a second order matrix differential equation. The study of a Rodrigues' formula for this new class, for the case of an arbitrary N remains open.

1.5 Motivation

1.5.1 The Bochner problem

In 1929 S. Bochner [19] posed and solved the following problem (also considered before by E. Routh in [151]):

Problem 1. *[The Bochner' problem] Consider the second order linear differential equation*

$$\rho_2 y^{''} + \rho_1 y^{'} + \rho_0 y = \lambda y, \tag{1.31}$$

where the coefficients $\rho_j, j = 0, 1, 2$ are assumed to be real-valued smooth functions on an interval I of the real line with $\rho_2 \neq 0$ and λ is a real parameter. When are there a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers and a system of polynomials $\{P_n\}_{n=0}^{\infty}$ such that (1.31) *holds?.*

Bochner proved that there are essentially (that is, up to a linear change of variable) only four distinct orthogonal polynomial sets satisfying the differential equation (1.31). They are called the classical orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel, which were already introduced in Section 1.1. He also implicitly imposed the problem of classifying all orthogonal polynomials satisfying linear differential equations of arbitrary order.

Krall in [91] found a remarkable theorem (cf. Theorem 2.1) characterizing all differential equations of the form

$$\sum_{k=1}^{M} \rho_k y^{(k)} = \lambda y, \tag{1.32}$$

where the coefficients $\rho_j, j = 1, \ldots, M$ are assumed to be real-valued smooth functions on an interval I of the real line with $\rho_M \neq 0$ and λ is a real parameter for which there exists a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of real numbers and a system of orthogonal polynomials $\{P_k\}_{k=0}^{\infty}$ satisfying (1.32). His proof in [91] is based on the notion of a dual equation to the differential equation (1.32), which is developed by Sheffer [156]. Later on, a simpler proof using the generating functions of orthogonal polynomials was found by Krall and Sheffer [93].

The classifying problem itself is not solved yet in general except for second order differential equations (due to Bochner [19]) and for fourth order differential equations (due to Krall [92]). We refer here the excellent publications, [62, 63] on the state of the art of the past two decades of the classes of orthogonal polynomials which are solutions of (1.32), for some fixed M.

There exists a third proof of Krall's theorem given in [98] based on the fact that if the differential equation (1.32) has orthogonal polynomial solutions then it must be *symmetrizable on polynomials*. This property will be used in Chapter 4 to give a necessary and sufficient condition that characterizes some classes of operators for which the relation of orthogonality (1.9) gives an inner product.

Systems of orthogonal polynomials which simultaneously are orthogonal with respect to some positive measure and solve the Bochner's problem, for some M, are called *Bochner-Krall* systems.

Motivated by quantum mechanics, in [165] and [166] Turbiner introduced the exactly solvable linear differential operators as the solution to the problem of a general classification of linear differential operators having a certain number of eigenfunctions. This problem is referred to as the *generalized Bochner's problem* (in the sense of Turbiner), see [166], more precisely

DEFINITION 1.7. *Let us give the name of generalized Bochner's problem to the problem of classifying the linear differential operators $\mathcal{L}^{(M)}[y] = \sum_{k=1}^{M} \rho_k y^{(k)}$ having $n+1$ polynomial eigenfunctions of the degree not higher than $n+1$.*

The class of operators which solve the Bochner's problem are termed *quasi-exactly-solvable* or *exactly-solvable* according to the case if n is finite or infinite respectively, that is

DEFINITION 1.8. *We will say that a linear differential operator of the M-th order, $\mathcal{L}^{(M)}$ is quasi-exactly-solvable, if $\mathcal{L}^{(M)}[\mathbb{P}_n] = \mathbb{P}_n$ for some n. Correspondingly, the operator $\mathcal{L}^{(M)}$, for which $\mathcal{L}^{(M)}[\mathbb{P}_n] \subseteq \mathbb{P}_n$, for all n, with equality for infinitely many indexes is named exactly-solvable.*

According to [166, Th. 1], [167, Th. 1.1], the generalized Bochner's problem has $n+1$ eigenfunctions in the form of a polynomial of order not greater than n if and only if the operator is quasi-exactly solvable and the same problem has an infinite sequence of polynomial eigenfunctions if and only if the operator is exactly solvable.

It is easy to see that if the differential operator $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \dfrac{d^k}{dx^k}$ has polynomial solutions P_n of degree n for $n = 0, 1, \ldots, M$ then the coefficients $\rho_j, j = 1, \ldots, M$ of $\mathcal{L}^{(M)}$ must be polynomials of degree at most j with equality for at least one index. They split into two major classes: non-degenerate and degenerate, where in the former case $deg[\rho_M] = M$ and in the latter case $deg[\rho_M] < M$.

Properties of the zeros of eigenpolynomials of these class of operators have been studied in detail in [132, 15, 16, 17]. Supported mainly in these results, we study in Chapter 5 the strong asymptotic behavior of the sequence of polynomial eigenfunctions of a certain class of these operators.

From the point of view of the theory of differential operators, the eigenfunctions of the class of exactly solvable are the most studied and this motivate us to study of the families of orthogonal polynomials with respect to exactly solvable operators and this is the objective of Chapter 4.

Bochner's problem is also a particular case of the following problem

Problem 2. *[The matrix-valued Bochner problem] Find all nontrivial matrix polynomials sequences $\{P_n\}_{n=0}^{\infty}$ of size $N \times N$ satisfying the following equations:*

$$D_{n+1}P_{n+1}(t) + E_nP_n(t) + D_{n-1}^{*}P_{n-1}(t) = zP_n(t), \tag{1.33}$$

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0(t) = \Lambda_n P_n(t), \tag{1.34}$$

where $E_n = E_n^{*}, det[D_n] \neq 0$ are matrices of size $N \times N$, F_2, F_1, F_0 are $N \times N$ matrix valued differentiable functions on an interval of the real line and Λ_n are Hermitian matrices.

In virtue of [9], we see that condition (1.33) is equivalent to say that the matrix polynomials $\{P_n\}_{n=0}^{\infty}$ are orthonormal with respect to some Hermitian matrix of measures W which is positive definite. An approach in the quest for the solution of this problem is taken by Durán in [45], where the problem of characterizing those positive definite matrix weights whose matrix orthogonal polynomials are solutions of some second order differential equation with matrix coefficients is raised, see also Section 1.4.2 for more details. As as already mentioned, all the families of polynomials solving the classical Bochner problem are known. The search of the families for the matrix valued case still remains open. In Chapter 6 we find a new class of matrix polynomials solving Problem 2.

We remark that Problem 2 is also part of a big family of problems known as *bispectral problems*, see the contribution of [75]. The consideration of Problem 2 as a bispectral one is the other approach for the quest of the solution of Problem 2, see the contributions [77, 79, 80, 144]. The reader interested in a general discussion of the bispectral problem can consult [10, 40, 85, 168].

1.5.2 Galerkin's method

Our main motivation starts from the work [8], where the authors introduce the concept of orthogonal polynomials with respect to a differential operator. This kind of orthogonality is relatively new and is mainly motivated by the advances in the last 20 years of the Sobolev type orthogonality and arises in a natural way from problems in approximation theory and mathematical physics. This concept, as we shall show, turns out to be in connection with Galerkin's method and fluids mechanics.

Galerkin's method is one of the most used and effective numerical methods in problems in which it is required to determine an unknown function that solves an linear equation (see[90, §9.4]). We will start formally by introducing one of its variants, assume that $\mathcal{L}[f] = g$, where $\mathcal{L}$ is a linear operator (linear differential ordinary in our case), g is a given function and f is a function to be determined from the equation.

Let $f_0,\ f_1,\ \ldots,\ f_n$, $n \in \mathbb{Z}_+$, be a set of *basic* or *test functions*, such that $f = \sum_{k=0}^{n} c_k f_k$ where $c_0, c_1, \ldots, c_n$ are scalars, by linearity

$$\mathcal{L}[f] = \sum_{k=0}^{n} c_k\, \mathcal{L}[f_k] = g. \tag{1.35}$$

Since the operator $\mathcal{L}$ and the functions $g,\ f_0,\ f_1,\ \ldots,\ f_n$ are known, we can consider (1.35) as a linear system of equations in the unknowns $c_0, c_1, \ldots, c_n$. We can expect that the system is inconsistent, because g will not generally lie in E_k, the vector space spanned by the functions $\mathcal{L}[f_0], \mathcal{L}[f_1], \ldots, \mathcal{L}[f_n]$. We therefore solve (1.35) approximately and obtain thereby an *approximate solution of the equation* $\mathcal{L}[f] = g$.

The approximate solution of (1.35) can be carried out according to many different criteria, each of which leads to a different approximate solution f. One of the most natural approach is to select $c_0, c_1, \ldots, c_k$ such that $h = \sum_{k=0}^{n} c_k\, \mathcal{L}[f_n]$ is *the element of best approximation* to g from E_k with respect to a suitable norm $\|\cdot\|$, i.e.

$$\|h - g\| = \min_{c_0,\ldots,c_n} \left\| \sum_{k=0}^{n} c_k\, \mathcal{L}[f_k] - g \right\|. \tag{1.36}$$

This is a problem in best approximation. We are approximating g by the nearest element in the subspace generated by the functions $\{\mathcal{L}[f_k]\}$.

A very general way of obtaining an approximate solution of Equation (1.35) is to select a set of linear functionals $\Lambda_1, \ldots, \Lambda_n$ and to impose the condition

$$\Lambda_i \left[\sum_{k=0}^{n} c_k\, \mathcal{L}[f_k] - g \right] = 0, \quad 0 \leq i \leq n.$$

By the linearity of the functionals, this becomes

$$\Lambda_i \left[\sum_{k=0}^{n} c_k\, \mathcal{L}[f_k] \right] = \Lambda_i\,[g]\,, \quad 0 \leq i \leq n. \tag{1.37}$$

Equation (1.37) is a system of k linear equations in the k unknowns c_k. If the functionals are point–evaluation functionals, defined by

$$\Lambda_i[v] = v(x_i), \quad 0 \leq i \leq n$$

then the method outlined above is called *a collocation method*. Galerkin methods are usually referred to as all the manifestations of the preceding strategy, see for example [32]. The *classical Galerkin method* is a particular case of Equation (1.37) in a Hilbert space, where $\Lambda_i[v] = \langle f_i, v\rangle$. Thus the equations to be solved in this case are

$$\sum_{k=0}^{n} c_k \langle f_i, \mathcal{L}[f_k]\rangle = \langle f_i, g\rangle, \quad 0 \leq i \leq n.$$

Now, let μ be a positive Borel measure supported on $\Delta \subset \mathbb{R}$ and $\{P_k\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to μ, $g \in L^2(\mu)$ and suppose that $\mathcal{L}$ is a linear ordinary differential operator with polynomial coefficients such that $\mathcal{L}[f]$ and f are polynomials of the same degree and such that $\mathcal{L}$ and μ satisfy the conditions of Definition (1.4). If $\|\cdot\|_\mu$ is the L^2-norm associated to μ on Δ, then *the element of the best approximation*, see [90, §6.8], to g from $E_n = \mathbb{P}_n$ is

$$h = \sum_{k=0}^{n} c_k\, P_k, \quad c_k = \frac{\int g\, P_k\, d\mu}{\int P_k^2\, d\mu}.$$

Naturally the following questions arise: Does there exist a set of monic polynomial *basic* or *test functions* $\{f_k\}_{k=0}^n$, where $deg[f_k] = k$, such that $\mathcal{L}[f_k] = \lambda_k P_k$, for some suitable constant λ_k and for every $k, n \in \mathbb{Z}_+$?, which are their properties?.

It is not difficult to see from Definition (1.4) that f_k is the monic orthogonal polynomial of degree k with respect to the pair $(\mathcal{L}, \mu)$ associated to the index k. We recall that the existence of the sequence $\{f_k\}_{k=0}^n$ can not always be guaranteed, let us say, for a general measure μ or operator $\mathcal{L}$ of the class defined above. Theorems 4.3, 4.4 and 4.6 of Chapter 4 of this thesis deals with such conditions of existence and uniqueness. Notice also that f_k can be characterized as the solution of the extremal problem $\|\mathcal{L}[f_k]\|_\mu = \min_{h_k(x)=x^k+\cdots} \|\mathcal{L}(h_k)\|_\mu$. It would be an interesting problem to study the properties of this set of polynomial test functions from the Numerical Analysis point of view and to consider also the relation of the orthogonality with respect to a differential operator to the Galerkin method for the general case of a class of operators satisfying the conditions of Definition (1.4), but we shall not deal with it in this thesis.

The study of strong asymptotic properties of eigenpolynomials of exactly solvable operators and the consideration of matrix orthogonal polynomials arose from the Bochner problem, which were already discussed in Section 1.5.1.

We finally mention the hydrodynamical (electrostatic equivalently) model for the zeros of orthogonal polynomials with respect to a differential operators. This work starts from [13, 12], where the authors find an hydrodynamical interpretation of the zeros of the Gegenbauer polar polynomials, already introduced in Section 1.3.2. In Chapters 2 and 3 we show that for the case of a Jacobi, Laguerre or Hermite operator, generically denoted by $\mathcal{L}_c$, it is also possible to obtain hydrodynamical interpretation of the zeros of the sequence of orthogonal polynomials with respect to $(\mathcal{L}_c, \mu)$, where μ is a measure satisfying certain conditions.

1.6 Structure of the thesis

This thesis deals with the concept of orthogonal polynomials with respect to a differential operator, the study of the strong asymptotic behavior of eigenpolynomials of exactly solvable operators, and matrix orthogonal polynomials. We have divided this work in seven chapters.

In Chapter 1 we introduce some concepts and background of the general theory of orthogonal polynomials as well as the state of the art of the theory that precede the results of this thesis.

The aim of Chapter 2 is the study of orthogonal polynomials with respect to a Jacobi operator

$$\mathcal{L}^{(\alpha,\beta)}[f](x) = (1-x^2)f''(x) + (\beta - \alpha - (\alpha+\beta+2)x)f'(x), \alpha, \beta > -1, f \in \mathbb{P},$$

and a finite positive Borel measure μ on $[-1, 1]$ satisfying certain conditions. For a positive integer m, we study the conditions over the measure μ in order to guarantee the existence of an infinite sequence of monic polynomials $\{Q_n\}_{n=m+1}^{\infty}$, where $deg[Q_n] = n$, satisfying the condition

$$\int \mathcal{L}^{(\alpha,\beta)}[Q_n](x)x^k d\mu(x) = 0 \quad \text{for all} \quad 0 \le k \le n-1.$$

We consider algebraic and analytic properties of this sequence. A fluid dynamics model for the interpretation of the zeros of these polynomials is also considered.

In Chapter 3, we deal with the case of orthogonality with respect to either a Laguerre or Hermite operator. We show the existence of recurrence relations for orthogonal polynomials with respect to these class of operators and for the derivatives as well. As in the case of a Jacobi operator considered in Chapter 2, the zeros of these polynomials and the zeros of the derivatives have an interpretation in terms of a fluid dynamical model. We study also the asymptotic properties of these polynomials by scaling with an appropriate parameter.

In Chapter 4 we generalize the results of Chapters 2 and 3 by considering orthogonal polynomials with respect to a linear differential operator

$$\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k},$$

where $\{\rho_k\}_{k=0}^{M}$ are complex polynomials such that $deg[\rho_k] \leq k, 0 \leq k \leq M$, with equality for at least one index. We analyze the uniqueness and zero location of these polynomials. An interesting phenomena occurring in this kind of orthogonality is the existence of operators for which the associated sequence of orthogonal polynomials reduces to a finite set. For a given operator we also find a classification, in terms of a system of difference equations with varying coefficients, of the measures for which it is possible to guarantee the existence of an infinite sequence of orthogonal polynomials. We also obtain a curve which contains the set of accumulation points of the zeros of these polynomials for the case of a first order differential operator giving also the strong asymptotic behavior. The results of this chapter have been submitted for consider for publication in [22].

In Chapter 5 we study the strong asymptotic behavior the eigenpolynomials of exactly solvable operators $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k}$. The study of the strong asymptotic behavior have drawn a great deal of attention in connection with problems of the theory of orthogonal polynomials and approximation theory. Some properties of the eigenpolynomials of these class of operators have been previously studied in [132] for operators of the form $\mathcal{L}^{(M)}[f](z) = \frac{d^M}{dz^M}\left(H_M(z)f(z)\right)$, where H_M is a fixed polynomial of degree M, and for exactly solvable operators by [15], [16] and [17]. Under the assumption that ρ_M is real, we obtain a formula for the strong asymptotic behavior of the eigenpolynomials of $\mathcal{L}^{(M)}$ on certain compact subsets of $\mathbb{C}$.

As an application, we consider the sequence of monic orthogonal polynomials with respect to the Sobolev inner product,

$$\langle P, Q\rangle = P(1)\overline{Q}(1) + \mu\, P'(1)\overline{Q}'(1) + \int_{-1}^{1} P'\overline{Q}' dx, \quad P, Q \in \mathbb{P}, \quad \mu > 0$$

which are eigenfunctions of the fourth order differential operator, cf.[89]

$$\mathcal{L}^{(M)}[u] = (z^2-1)^2 u^{(4)} + 4z(z^2-1)u^{(3)} + 2(z-1)((1+2\,A)z + 2\,A + 3)u^{(2)},$$

we obtain a formula for the strong asymptotic behavior of this sequence for compact subsets of $\mathbb{C} \setminus [-1, 1]$. The results of this chapter have been submitted for consider for publication in [25].

Chapter 6 concerns with matrix orthogonal polynomials. We find a new class of matrix orthogonal polynomials of arbitrary order satisfying a second order matrix differential equation. For matrix polynomials of size 2×2, we find an explicit expression for the sequence of orthonormal polynomials with respect to a matrix weight by using a Rodrigues' formula for these polynomials. In particular, we show that one of the recurrence coefficients for this sequence of orthonormal polynomials does not asymptotically behave as a scalar multiple of the identity, as it happens in the examples studied up to now in the literature. The results of this chapter have been published in [21].

In Chapter 7 are given a summary of the work and some open problems.

Chapter 2

Orthogonal polynomials with respect to a Jacobi operator

2.1 Introduction

In this chapter we consider orthogonal polynomials with respect to a Jacobi operator. Let μ be a finite positive Borel measure on $[-1,1]$ and $\{L_n\}_{n=0}^{\infty}$ the corresponding system of monic orthogonal polynomials; i.e. $L_n(x) = x^n + \dots$ and

$$\langle L_n, L_k \rangle_\mu = \int L_n(x) L_k(x) d\mu(x) \begin{cases} \neq 0 & \text{if } n = k, \\ = 0 & \text{if } n \neq k. \end{cases} \tag{2.1}$$

Denote by $\mathcal{L}^{(\alpha,\beta)}$ the Jacobi differential operator on the space $\mathbb{P}$, with $\alpha, \beta > -1$, where

$$\mathcal{L}^{(\alpha,\beta)}[f] = (1-x^2)f'' + (\beta - \alpha - (\alpha+\beta+2)x)f', \quad f \in \mathbb{P}, \tag{2.2}$$

or equivalently (cf. [163, (4.2.2)])

$$\mathcal{L}[f] = \frac{\left((1-x)^{\alpha+1}(1+x)^{\beta+1} f'\right)'}{(1-x)^{\alpha}(1+x)^{\beta}}, \quad f \in \mathbb{P}. \tag{2.3}$$

From (2.2) it follows that f and $\mathcal{L}[f]$ are polynomials of the same degree. It is straightforward that integrating (2.3) with respect to the (α,β)-Jacobi measure $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$ on $[-1,1]$, we obtain

$$\int \mathcal{L}^{(\alpha,\beta)}[f](x)\,d\mu_{\alpha,\beta}(x) = 0, \quad f \in \mathbb{P}. \tag{2.4}$$

We say that Q_n is the nth monic orthogonal polynomial with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ if $\deg[Q_n] \leq n$ and

$$\int \mathcal{L}^{(\alpha,\beta)}[Q_n](x) x^k d\mu(x) = 0 \quad \text{for all} \quad 0 \leq k \leq n-1. \tag{2.5}$$

This nonstandard orthogonality with respect to a differential operator was introduced in [8], where the authors prove conditions of existence and uniqueness. The starting points of this work are [12, 13], where the orthogonality with respect to the differential operator $\mathcal{L}_\zeta[f](z) = f(z) + (z-\zeta)f'(z)$, $\zeta \in \mathbb{C}$, was analyzed.

From (2.1), we have that a monic polynomial Q_n of degree n is orthogonal with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ if and only if it is a polynomial solution of the differential equation

$$\mathcal{L}^{(\alpha,\beta)}[Q_n] = \lambda_n L_n, \quad \text{where} \quad \lambda_n = \lambda_n^{(\alpha,\beta)} = -n(1+n+\alpha+\beta). \tag{2.6}$$

As will be shown, it is not always possible to guarantee the existence of a system of polynomials $(Q_n)_{n\in\mathbb{Z}_+}$ orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$. Let $\alpha,\beta>-1$ and $m\in\mathbb{N}$ be fixed. A fundamental role in this chapter is played by the class $\mathcal{P}_m(\alpha,\beta)$ defined as the family of finite positive Borel measures μ supported on $[-1,1]$ for which there exist a non negative polynomial ρ of degree m, such that $d\mu(x)=\dfrac{1}{\rho(x)}d\mu_{\alpha,\beta}(x)$.

This chapter deals with some algebraic and analytic aspects of the sequence of orthogonal polynomials $(Q_n)_{n=m+1}^{\infty}$ orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$. We provide asymptotic results for the sequence of the orthogonal polynomials with respect to $(\mathcal{L}^{(\alpha,\beta)},\mu)$ and study the set of accumulation points of their zeros as well. In particular, we prove

THEOREM 2.1. *Let $\mu\in\mathcal{P}_m(\alpha,\beta)$, where $m\in\mathbb{N}$ and $\alpha,\beta>-1$. If $(\zeta_n)_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta\in\mathbb{C}\setminus[-1,1]$ and $(Q_n)_{n=m+1}^{\infty}$ the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$ such that $Q_n(\zeta_n)=0$, then the accumulation points of zeros of $(Q_n)_{n=m+1}^{\infty}$ are located on the set $E=\mathcal{E}(\zeta)\bigcup[-1,1]$, where $\mathcal{E}(\zeta)$ is the ellipse*

$$\mathcal{E}(\zeta):=\{z\in\mathbb{C}: z=\cosh(\eta_\zeta+i\theta), 0\le\theta<2\pi\}, \tag{2.7}$$

and $\eta_\zeta:=\ln|\varphi(\zeta)|=\ln|\zeta+\sqrt{\zeta^2-1}|$. If $\delta(\zeta)=\inf\limits_{-1\le x\le 1}|\zeta-x|>2$ then $E=\mathcal{E}(\zeta)$.

Let $\varphi(z)=z+\sqrt{z^2-1}$ be the function which maps the complement of $[-1,1]$ onto the exterior of the unit circle, where we take the branch of $\sqrt{z^2-1}$ for which $|\varphi(z)|>1$ whenever $z\in\mathbb{C}\setminus[-1,1]$.

If $\mu\in\mathcal{P}_m(\alpha,\beta)$, where $\alpha,\beta>-1$ and $m\in\mathbb{N}$, let $\nu_1,\nu_2,\dots,\nu_m\in\mathbb{C}$ denote the m zeros of the polynomial $\rho(z)=r\prod_{i=1}^{m}(z-\nu_i)$, such that $d\mu_{\alpha,\beta}(x)=\rho(x)d\mu(x)$.

For all $z\in\mathbb{C}\setminus[-1,1]$ we define the function $\Phi(\rho,z)$ and a constant ϕ_m as

$$\Phi(\rho,z)=\prod_{k=1}^{m}\frac{z-\nu_i}{\varphi(z)-\varphi(\nu_i)},\quad \phi_m=2^m\,\exp\left(\frac{1}{2\pi}\int\frac{\log(\rho(t))}{\sqrt{1-t^2}}dt\right).$$

The function $\phi_m\,\Phi(\rho,z)$ is a particular case of the well known *Szegő's function* (cf. [137, §6.1]).

For the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$ the following asymptotic behavior holds

THEOREM 2.2. *Let $\{\zeta_n\}_{n=m+1}^{\infty}$ be a sequence of complex numbers with limit $\zeta\in\mathbb{C}\setminus[-1,1]$, $m\in\mathbb{N}$, $\mu\in\mathcal{P}_m(\alpha,\beta)$, where $\alpha,\beta>-1$ and $\{Q_n\}_{n=m+1}^{\infty}$ the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$ such that $Q_n(\zeta_n)=0$, then:*

1. *Uniformly on compact subsets of $\Omega=\{z\in\mathbb{C}:|\varphi(z)|>|\varphi(\zeta)|\}$, (i.e. the exterior of the ellipse $\mathcal{E}(\zeta)$)*

$$\frac{Q_n(z)}{P_n^{(\alpha,\beta)}(z)}\ \underset{n}{\rightrightarrows}\ \phi_m\,\Phi(\rho,z). \tag{2.8}$$

2. *Uniformly on compact subsets of $\Omega=\{z\in\mathbb{C}:|\varphi(z)|<|\varphi(\zeta)|\}\setminus[-1,1]$*

$$\frac{Q_n(z)}{P_n^{(\alpha,\beta)}(\zeta_n)}\ \underset{n}{\rightrightarrows}\ -\phi_m\,\Phi(\rho,\zeta). \tag{2.9}$$

If $\delta(\zeta)>2$ then (2.9) *holds for $\Omega=\{z\in\mathbb{C}:|\varphi(z)|<|\varphi(\zeta)|\}$ (i.e. the interior of the ellipse $\mathcal{E}(\zeta)$).*

The chapter is organized as follows. In Section 2.2 we study the existence of a system of polynomials $\{Q_n\}_{n=m+1}^{\infty}$, orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$. Section 2.3 is devoted to the study of recurrence relations and location of zeros of the polynomials . In Sections 2.4 and 2.5 we study the asymptotic behavior of the orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)},\mu)$ and its zeros respectively. In the last section, we introduce a fluid dynamics model for the interpretation of the critical points of Q_n. The results of this chapter have been submitted for consider for publication in [23].

2.2 Existence and uniqueness

It is well known that the differential operator $\mathcal{L}^{(\alpha,\beta)}$ has a system of monic eigenpolynomials $\left\{P_n^{(\alpha,\beta)}\right\}_{n=0}^{\infty}$ and a sequence of constant $\{\lambda_n\}_{n=0}^{\infty}$ (eigenvalues), such that

$$\mathcal{L}^{(\alpha,\beta)}[P_n^{(\alpha,\beta)}] = \lambda_n P_n^{(\alpha,\beta)}, \tag{2.10}$$

where the eigenpolynomial $P_n^{(\alpha,\beta)}$ is the nth monic orthogonal polynomial with respect to the (α,β)-measure of Jacobi $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}\,(1+x)^{\beta}\,dx$ on $[-1,1]$, with $\alpha,\beta > -1$, i.e.

$$\left\langle P_n^{(\alpha,\beta)}, x^k \right\rangle_{\alpha,\beta} = \int P_n^{(\alpha,\beta)}(x)\, x^k d\mu_{\alpha,\beta}(x) = 0, \text{ for all } 0 \le k \le n-1. \tag{2.11}$$

Furthermore, from [163, (4.21.6) and (4.3.3)]

$$\begin{aligned} \tau_n &= \left\langle P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)} \right\rangle_{\alpha,\beta} \\ &= \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{-(2n+\alpha+\beta+1)}\,\Gamma(2n+\alpha+\beta+2)\,\Gamma(2n+\alpha+\beta+1)}. \end{aligned} \tag{2.12}$$

THEOREM 2.3. *Let n be a fixed natural number and μ a finite positive Borel measure on $[-1,1]$. Then, the differential equation* (2.6) *has a monic polynomial solution Q_n of degree n, which is unique up to an additive constant, if and only if*

$$\int L_n(x) d\mu_{\alpha,\beta}(x) = 0. \tag{2.13}$$

where L_n is the nth monic orthogonal polynomials with respect to the measure μ.

Proof.

Suppose that there exists a polynomial Q_n of degree n, such that $\mathcal{L}^{(\alpha,\beta)}[Q_n] = -n(1+n+\alpha+\beta)\,L_n$, where L_n is the nth monic orthogonal polynomial for μ. From the orthogonality of the Jacobi polynomials

$$Q_n(z) = P_n^{(\alpha,\beta)}(z) + \sum_{k=0}^{n-1} a_{n,k} P_k^{(\alpha,\beta)}(z), \tag{2.14}$$

$$L_n(z) = P_n^{(\alpha,\beta)}(z) + \sum_{k=0}^{n-1} b_{n,k} P_k^{(\alpha,\beta)}(z), \tag{2.15}$$

where $a_{n,k} = \frac{1}{\tau_k}\left\langle Q_n, P_k^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}$ and $b_{n,k} = \frac{1}{\tau_k}\left\langle L_n, P_k^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}$.

Replacing Q_n and L_n in (2.6) by the linear combinations (2.14) and (2.15), from the linearity of $\mathcal{L}^{(\alpha,\beta)}[\cdot]$ and (2.10) we get

$$b_{(n,0)} = \frac{1}{\mu_{\alpha,\beta}([-1,1])} \int L_n(x) d\mu_{\alpha,\beta}(x) = 0.$$

Conversely, suppose that L_n, the nth monic orthogonal polynomial with respect to μ satisfies (2.13). Let Q_n be the polynomial of degree n defined by

$$Q_n(z) = P_n^{(\alpha,\beta)}(z) + \sum_{k=0}^{n-1} a_{n,k} P_k^{(\alpha,\beta)}(z), \tag{2.16}$$

where $a_{(n,0)}$ is an arbitrary constant and

$$a_{n,k} = \frac{\lambda_n}{\lambda_k}\, b_{n,k} = \frac{\lambda_n}{\lambda_k\,\tau_k} \left\langle L_n, P_k^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}, \quad k = 1, \ldots, n-1.$$

From the linearity of $\mathcal{L}^{(\alpha,\beta)}[\cdot]$ and (2.10) we get that $\mathcal{L}^{(\alpha,\beta)}[Q_n] = -n(1+n+\alpha+\beta)\, L_n$. □

We are interested in discussing systems of polynomials such that for all n sufficiently large they are solutions of (2.6). In this sense, the next corollary is fundamental.

COROLLARY 2.1. *Let μ be a finite positive Borel measure on $[-1,1]$ such that $d\mu(x) = r(x)d\mu_{\alpha,\beta}(x)$, where $r \in L^2(\mu_{\alpha,\beta})$. Then, m is the smallest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$, if and only if r^{-1} is a polynomial of degree m.*

Proof. Suppose that m is the smallest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$. According to Theorem 2.3

$$\int \frac{1}{r(x)} L_n(x) d\mu(x) = \int L_n(x) d\mu_{\alpha,\beta}(x) \begin{cases} = 0 & \text{if } n > m \\ \neq 0 & \text{if } n = m \end{cases}.$$

But this is equivalent to saying that $\dfrac{1}{r(x)} = \displaystyle\sum_{k=0}^{m} c_k L_k(x)$ with $c_m \neq 0$. The converse is straightforward. □

From the previous corollary, if $\mu \in \mathcal{P}_m(\alpha,\beta)$ then the differential equation (2.6) has an unique monic polynomial solution Q_n of degree n for all $n > m$, except for an additive constant.

Let $\{\zeta_n\}_{n=m+1}^{\infty}$ be a sequence of complex numbers, where $m \in \mathbb{N}$ is fixed, and assume that $\mu \in \mathcal{P}_m(\alpha,\beta)$. We complement the definition of the sequence $\{Q_n\}_{n=m+1}^{\infty}$ in (2.5) by considering that henceforth Q_n for each $n > m$ is the polynomial solution of the initial value problem

$$\begin{cases} \mathcal{L}^{(\alpha,\beta)}[y] &= \lambda_n L_n, \quad n > m, \\ y(\zeta_n) &= 0. \end{cases} \tag{2.17}$$

Then, we say that $\{Q_n\}_{n=m+1}^{\infty}$, is the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ such that $Q_n(\zeta_n) = 0$.

If $\widehat{Q}_n$ is the monic polynomial of degree n defined by the formula

$$\widehat{Q}_n(z) = \lambda_n \sum_{k=0}^{m} \frac{b_{n,n-k}}{\lambda_{n-k}} P_{n-k}^{(\alpha,\beta)}(z), \quad b_{n,n-k} = \frac{1}{\tau_{n-k}} \left\langle L_n, P_{n-k}^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}; \tag{2.18}$$

then, the initial value problem (2.17) has the unique polynomial solution

$$y(z) = Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n). \tag{2.19}$$

2.3 The polynomial Q_n'

Let $m \in \mathbb{N}$, $\{\zeta_n\}_{n=0}^{\infty}$ a sequence of complex numbers, and $\mu \in \mathcal{P}_m(\alpha,\beta)$ be fixed, then for all $n > m$ the polynomials Q_n (solution of (2.17)) are uniquely determined by (2.18)–(2.19). Without loss of generality, we will complete the sequence of polynomials Q_n for all $n \in \mathbb{N}$ as follows

$$\begin{aligned} Q_n(z) &= \left(P_n^{(\alpha,\beta)}(z) - P_n^{(\alpha,\beta)}(\zeta_n)\right) \\ &\quad + \lambda_n \sum_{k=1}^{\min(m,n)} \frac{b_{n,n-k}}{\lambda_{n-k}} \left(P_{n-k}^{(\alpha,\beta)}(z) - P_{n-k}^{(\alpha,\beta)}(\zeta_n)\right), n \geq 1 \\ Q_0(z) &= 1 \end{aligned} \tag{2.20}$$

For convenience, only in the previous formula we consider $\lambda_0 = 1$. Note that $\{Q_n\}_{n=0}^{\infty}$ defined by (2.20) is a system of polynomials, such that $Q_n(\zeta_n) = 0$ for all $n \geq 1$. Let us remark that if $n \leq m$, in general, $\mathcal{L}^{(\alpha,\beta)}[Q_n] \neq \lambda_n L_n$.

Additionally, as the degree of a polynomial is invariant under the operator $\mathcal{L}^{(\alpha,\beta)}[\cdot]$ and the polynomial Q_n, for all $n \leq m$, is of degree n (see (2.20)),

$$\{1, \mathcal{L}^{(\alpha,\beta)}[Q_1], \ldots, \mathcal{L}^{(\alpha,\beta)}[Q_m], L_{m+1}, \ldots, L_n, \ldots\} \tag{2.21}$$

is a polynomial system.

In the following result, we show that for $n > (2m+1)$ the derivatives of the system of polynomials Q_n satisfy a recurrence relation with a fixed finite number of terms.

THEOREM 2.4. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\alpha, \beta)$, where $\alpha, \beta > -1$. Then if R is any primitive of ρ, for each $n > (2m+1)$ the sequence of polynomials Q'_n satisfies the relation*

$$R(z)Q'_n(z) = \sum_{k=-m-1}^{m+1} \theta_{R,n,n-k}\, Q'_{n-k}(z), \tag{2.22}$$

where the initial values $Q'_{m+1}, \ldots, Q'_{2m+2}$ are given by the derivatives of (2.18) *and*

$$\begin{aligned} \theta_{R,n,n-k} &= \frac{1}{\lambda_{n-k}} (\lambda_n\, e_{R,n,n-k} + d_{n,n-k}), \\ e_{R,n,n-k} &= \frac{1}{l_{n-k}} \int R(x)\, L_{n-k}(x)\, L_n(x)\, d\mu(x), \\ l_i &= \int L_i^2(x) d\mu(x), \end{aligned} \tag{2.23}$$

$$\begin{aligned} d_{n,n-k} &= \frac{1}{l_{n-k}} \sum_{j=j_1(k)}^{j_2(k)} \tau_{n-j}\, \widetilde{c}_{n,n-j}\, b_{n-k,n-j}, \\ & \quad j_1(k) = \max\{-1, k\} \text{ and } j_2(k) = \min\{m+1, m+k\}. \\ \widetilde{c}_{n,n-k} &= \lambda_n \sum_{j=j_3(k)}^{j_4(k)} \frac{b_{n,n-j}\, c_{n-j,j-k}}{\lambda_{n-j}}, \\ & \quad j_3(k) = \max\{0, k-1\} \text{ and } j_4(k) = \min\{m, k+1\}, \\ c_{n,1} &= -n, \\ c_{n,0} &= \frac{2n(\alpha-\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \\ c_{n,-1} &= 4n \frac{(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)^2((2n+\alpha+\beta)^2-1)}. \end{aligned}$$

Before starting the proof of Theorem 2.4, we will state and prove some lemmas.

LEMMA 2.1. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\alpha, \beta)$, where $\alpha, \beta > -1$. Then for $n > m$*

$$L_n(z) = \sum_{k=0}^{m} b_{n,n-k}\, P_{n-k}^{(\alpha,\beta)}(z), \tag{2.24}$$

$$\rho(z) P_n^{(\alpha,\beta)}(z) = \tau_n \sum_{k=0}^{m} \frac{b_{n+k,n}}{l_{n+k}}\, L_{n+k}(z), \tag{2.25}$$

where $b_{i,j} = \frac{1}{\tau_j} \left\langle L_i, P_j^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}$ and τ_j as in (2.12).

Proof. As $\mu \in \mathcal{P}_m(\alpha,\beta)$ then $\langle L_n(x)\, x^k\rangle_{\alpha,\beta} = 0$ for all $n > m$ and $k < n-m$. Hence, $b_{n,k} = 0$ for $k = 0,1,\dots,n-m-1$ and (2.24) is established.

From the orthogonality relations (2.1) and (2.11), if $i < j$ or $i > j+m$, we have

$$\int L_i(x) P_j^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) = \int L_i(x) P_j^{(\alpha,\beta)}(x)\, \rho(x)\, d\mu(x) = 0. \tag{2.26}$$

The relation (2.25) is straightforward from the Fourier expansion of $\rho\, P_n^{(\alpha,\beta)}$ in terms of the $\{L_k\}$, $k = 0,1,\dots,n+m$ and (2.26). □

From the lemma above, the polynomials $\widehat{Q}_n$ defined in (2.18) and its derivatives can be written as a linear combination of the polynomials $\{L_n\}_{n=0}^{\infty}$ as we show in the next lema.

LEMMA 2.2. *Under the conditions of Lemma 2.1, for $n > m$ the polynomials $\widehat{Q}_n$ satisfy the following relations:*

$$\int \widehat{Q}_n(x)\, x^k\, d\mu_{\alpha,\beta}(x) = 0, \quad k = 0,1,\dots,n-m-1, \tag{2.27}$$

$$(1-z^2)\rho(z) Q_n'(z) = \sum_{k=-m-1}^{m+1} d_{(n-k,k)}\, L_{n-k}(z). \tag{2.28}$$

Proof. From (2.18) the relations (2.27) and

$$(1-z^2)\, Q_n'(z) = (1-z^2)\, \widehat{Q}_n'(z) = \lambda_n \sum_{k=0}^{m} \frac{b_{n,n-k}}{\lambda_{n-k}}\, (1-z^2) \left(P_{n-k}^{(\alpha,\beta)}(z)\right)', \tag{2.29}$$

follows directly.

Using the structure relation fulfilled by Jacobi polynomials (see [163, (4.5.5)–(4.5.6)]), we have

$$(1-z^2) \left(P_{n-k}^{(\alpha,\beta)}(z)\right)' = c_{n-k,1}\, P_{n-k+1}^{(\alpha,\beta)}(z) + c_{n-k,0}\, P_{n-k}^{(\alpha,\beta)}(z) + c_{n-k,-1}\, P_{n-k-1}^{(\alpha,\beta)}(z).$$

Substituting this formula into (2.29), we obtain

$$(1-z^2) Q_n'(z) = \sum_{k=-1}^{m+1} \widetilde{c}_{n,n-k}\, P_{n-k}^{(\alpha,\beta)}(z),$$

and from (2.25), (2.28) immediately follows. □

Proof of Theorem 2.4. As the sequence $\{Q_n\}_{n=0}^{\infty}$ is a system of polynomials, then the sequence of its derivatives $\{Q_n'\}_{n=0}^{\infty}$ is also system of polynomials. Hence, the polynomial $R\, Q_n'$ can be expanded as linear combination of the polynomials $\{\widehat{Q}_n'\}_{n=0}^{\infty}$, i.e. there exist $(n+m)$ constants $\theta_{(R,n,1)}, \dots, \theta_{(R,n,n+m)}$ such that

$$R(z) Q_n'(z) = \sum_{k=-m}^{n-1} \theta_{R,n,n-k}\, \widehat{Q}_{n-k}'(z). \tag{2.30}$$

Let $\widetilde{\mathcal{L}}^{(\alpha,\beta)}$ be the linear differential operator on the space of all polynomials $\mathbb{P}$ given by $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[f'] = \mathcal{L}^{(\alpha,\beta)}[f]$ for all $f \in \mathbb{P}$, i.e.

$$\widetilde{\mathcal{L}}^{(\alpha,\beta)}[f] = (1-x^2) f' + (\beta - \alpha - (\alpha+\beta+2)x) f.$$

Since $\{L_n\}_{n=0}^{\infty}$ is a system of polynomials and $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[\cdot]$ is a linear application, the polynomial $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[RQ'_n]$ can be written as a linear combination of the system of polynomials (2.21) as follows

$$\begin{aligned}\widetilde{\mathcal{L}}^{(\alpha,\beta)}\left[RQ'_n\right](z) &= \sum_{k=-m-1}^{n-m-1} \theta_{R,n,n-k}\,\lambda_{n-k}\,L_{n-k}(z) \\ &\quad + \sum_{k=n-m}^{n-1} \theta_{R,n,n-k}\,\mathcal{L}^{(\alpha,\beta)}\left[Q_{n-k}(z)\right], \end{aligned} \tag{2.31}$$

Taking $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[\cdot]$ on the left hand side of the equality (2.30), we get

$$\begin{aligned}\widetilde{\mathcal{L}}^{(\alpha,\beta)}[RQ'_n](z) &= R(z)\widetilde{\mathcal{L}}^{(\alpha,\beta)}[Q'_n(z)] + (1-z^2)\rho(z)Q'_n(z) \\ &= \lambda_n R(z)\,L_n(z) + (1-z^2)\rho(z)Q'_n(z).\end{aligned} \tag{2.32}$$

From (2.1)

$$R(z)\,L_n(z) = \sum_{k=-m-1}^{m+1} e_{R,n,n-k}\,L_{n-k}(z). \tag{2.33}$$

Substituting (2.28) and (2.33) in (2.32), we have

$$\widetilde{\mathcal{L}}^{(\alpha,\beta)}[RQ'_n](z) = \sum_{k=-m-1}^{m+1} \left(\lambda_n\,e_{R,n,n-k} + d_{n,n-k}\right)\,L_{n-k}(z). \tag{2.34}$$

As $n \geq 2(m+1)$, we can assume that (2.34) is the expansion of the polynomial $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[R\,Q'_n]$ in terms of the polynomials L_n. Now, identifying coefficients between (2.31) and (2.34) we have that $\theta_{R,n,n-k} = 0$ for all $k = 1,\dots,n-m-2$, and we have the formulas (2.22)–(2.23).

□

2.4 Asymptotic behavior of the polynomials $\widehat{Q}_n$ and their zeros

In this section we study the asymptotic behavior of the polynomials $\widehat{Q}_n$ and their zeros. The following result is essential in the proof of the theorems in this and the next section.

THEOREM 2.5. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha,\beta > -1$. Then*

$$\frac{\widehat{Q}_n(z)}{P_n^{(\alpha,\beta)}(z)} \underset{n}{\rightrightarrows} \phi_m^2\,\Phi(\rho,z), \tag{2.35}$$

uniformly on closed subsets of $\overline{\mathbb{C}} \setminus [-1,1]$ where $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$.

First, we state a preliminary lemma which follows from Theorems 26 and 29 of [137, §6.1].

LEMMA 2.3. *Let $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha,\beta > -1$ and $m \in \mathbb{N}$. If $\{L_n\}_{n=0}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to μ*

$$\frac{L_n(z)}{P_n^{(\alpha,\beta)}(z)} \underset{n}{\rightrightarrows} \phi_m^2\,\Phi(\rho,z),$$

uniformly on closed subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

Proof of Theorem 2.5 . From (2.18) and (2.24)

$$\frac{\widehat{Q}_n(z) - L_n(z)}{P_n^{(\alpha,\beta)}(z)} = \sum_{k=1}^{m} \left(\frac{\lambda_n}{\lambda_{n-k}} - 1 \right) b_{n,n-k} \, \frac{P_{n-k}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)}. \tag{2.36}$$

As $\lambda_n = -n(1+n+\alpha+\beta)$, for each k fixed, $k = 1, 2, \ldots, m$,

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-k}} = 1. \tag{2.37}$$

Let K be a closed subset of $\overline{\mathbb{C}} \setminus [-1,1]$. From the interlacing property of the zeros of consecutive Jacobi polynomials on $[-1,1]$, it easily follows that there exists a constant M_* such that for all $z \in K$

$$\left| \frac{P_{n-k}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)} \right| < M_k \leq M_*, \quad k = 1, \ldots, m, \tag{2.38}$$

where $M_k = \sup\limits_{\substack{z\in K\\ x\in[-1,1]}} |z-x|^{-k}$ and $M_* = \max\{M_1, M_m\}$.

From (2.12), it is not hard to see that there exist two monic polynomials of degree $4(m-k)$ in the variable n, $q_{1,4(m-k)}^{(\alpha,\beta)}(n)$ and $q_{2,4(m-k)}^{(\alpha,\beta)}(n)$, such that

$$\left\| P_{n-k}^{(\alpha,\beta)} \right\|_{\alpha,\beta}^2 = 4^{k-m} \frac{q_{1,4(m-k)}^{(\alpha,\beta)}(n)}{q_{2,4(m-k)}^{(\alpha,\beta)}(n)} \left\| P_{n-m}^{(\alpha,\beta)} \right\|_{\alpha,\beta}^2, \quad k = 1, 2, \ldots, m.$$

Therefore, from the Cauchy-Bunyakovsky-Schwarz inequality and the extremal property of the monic orthogonal polynomials, for n sufficiently large, we get

$$\begin{aligned} |b_{n,n-k}| &\leq \frac{\|L_n\|_{\alpha,\beta}}{\sqrt{\tau_{n-k}}} \leq \sqrt{\frac{c_1}{\tau_{n-k}}} \, \|L_n\|_\mu \leq \sqrt{\frac{c_1}{\tau_{n-k}}} \, \frac{\left\| \rho \, P_{n-m}^{(\alpha,\beta)} \right\|_\mu}{|r|} \\ &= 2^{m-k} \sqrt{\frac{c_1 \, q_{2,4(m-k)}^{(\alpha,\beta)}(n)}{\tau_{n-m} \, q_{1,4(m-k)}^{(\alpha,\beta)}(n)}} \, \frac{\left\| \rho \, P_{n-m}^{(\alpha,\beta)} \right\|_\mu}{|r|} \leq \frac{c_1 \, 2^m}{|r|} \end{aligned} \tag{2.39}$$

where $1 \leq k \leq m$, $c_1 = \sup\limits_{x\in[-1,1]} \rho(x)$, r is the leading coefficient of ρ. Hence by (2.36), (2.37), (2.38), and (2.39)

$$\left| \frac{\widehat{Q}_n(z) - L_n(z)}{P_n^{(\alpha,\beta)}(z)} \right| \underset{n}{\rightrightarrows} 0, \text{ uniformly on closed subsets of } \overline{\mathbb{C}} \setminus [-1,1],$$

and from Lemma 2.3 the asymptotic formula (2.35) is established.

□

COROLLARY 2.2. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha, \beta > -1$. Then*

1.

$$\lim_{n\to\infty} \left| \widehat{Q}_n(z) \right|^{\frac{1}{n}} = \frac{|z + \sqrt{z^2-1}|}{2}, \tag{2.40}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

2. *The set of accumulation points of the zeros of the sequence of polynomials* $\{\widehat{Q}_n\}_{n=m+1}^{\infty}$ *is* $[-1,1]$*, i.e.* $\bigcap_{n\geq 1}\bigcup_{k=n}^{\infty}\{z : \widehat{Q}_k(z)=0\} = [-1,1]$. *For each* n *at least* $(n-m)$ *zeros of* $\widehat{Q}_n$ *are contained on* $[-1,1]$.

Proof. The first part of the theorem is an immediate consequence of (2.35) and [163, (8.21.9) and (4.21.6)].

To prove the second part, from (2.27) it easily follows that $\widehat{Q}_n$ has at least $n-m$ zeros of $\widehat{Q}_n$ are contained in $[-1,1]$ (cf. [163, §3.3]).

The function in the right-hand side of (2.35) in Theorem 2.5 is holomorphic and does not have zeros in $\overline{\mathbb{C}}\setminus[-1,1]$. Let K be a closed subset of $\overline{\mathbb{C}}\setminus[-1,1]$, from the Rouché's theorem we have that for n large the polynomial $\widehat{Q}_n$ does not have zeros on K, i.e. the zeros of the sequence of polynomials $\{\widehat{Q}_n\}_{n=0}^{\infty}$ can not accumulate outside $[-1,1]$.

On the other hand, (2.40) implies the weak star asymptotic of the zero counting measures of the polynomials $\{\widehat{Q}_n\}_{n=m+1}^{\infty}$ (cf. [162, Ch. 2]). That is, if we associate to each $\widehat{Q}_n$ the measure $\mu_n = \dfrac{1}{n}\sum_{Q_n(\omega)=0}\delta_\omega$, then $d\mu_n(x) \overset{*}{\longrightarrow} \dfrac{1}{\pi}\dfrac{dx}{\sqrt{1-x^2}}$ (the equilibrium distribution on $[-1,1]$) in the weak-* topology and this implies that the zeros of $\{\widehat{Q}_n\}_{n=m+1}^{\infty}$ must be dense in $[-1,1]$.

□

2.5 Asymptotic behavior of the polynomials Q_n and their zeros

Some basic properties of the zeros of Q_n follow from (2.6). For example, the multiplicity of the zeros of Q_n is at most 3, a zero of multiplicity 3 is also a zero of L_n and a zero of multiplicity 2 is a critical point of $\widehat{Q}_n$.

From the second part of Corollary 2.2, we get that Q_n' has at least $(n-m-1)$ zeros of odd multiplicity on $]-1,1[$. For $m=1$ we have that

THEOREM 2.6. *Under the same hypothesis of Theorem 2.4, if* $m=1$ *the critical points of* Q_n *interlace the zeros of* L_n.

Proof. If $m=1$ then from (2.27) the polynomial $\widehat{Q}_n$ has at least $(n-1)$ real zeros of odd order on $]-1,1[$. But, $\widehat{Q}_n$ is a polynomial with real coefficients and degree n, consequently the zeros of $\widehat{Q}_n$ are real and simples. As $Q_n' = \widehat{Q}_n'$, from Rolle's theorem all the critical points of Q_n are real, simple and $(n-2)$ of them are contained on $]-1,1[$.

Denote $P(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}Q_n'(x)$. As α and β are real numbers in general P is not a polynomial. Notice that P is a real–valued differentiable function on $[-1,1]$. Without loss of generality, suppose that there exists $x\in]1,\infty[$ such that $P(x)=0$, as $P(1)=0$ from the Rolle's theorem there exists $x'\in]1,x[$ such that $P'(x')=0$. But, from (2.3) and (2.6) $P'(x) = \lambda_n(1-x)^{\alpha}(1+x)^{\beta}L_n(x)$ and all the critical points of P are contained in [-1,1]. Hence all the critical points of Q_n are contained in $]-1,1[$. Again using the Rolle's theorem, it is straightforward that the critical points of Q_n interlace the zeros of L_n.

□

From Corollary 2.2 we have that the set of accumulation points of Q_n' is $[-1,1]$. For $m=1$, Theorem 2.6 gives that the critical points of Q_n are in $[-1,1]$, interlace the zeros of L_n, and are simple. Numerical experiments also show this behavior for $m>1$. We conjecture that this always is the case.

For the proof of Theorems 2.1 and 2.2 we will use the following result.

LEMMA 2.4. *Let* $\mu\in\mathcal{P}_m(\alpha,\beta)$*, where* $m\in\mathbb{N}$ *and* $\alpha,\beta>-1$*. If* $\{\zeta_n\}_{n=m+1}^{\infty}$ *is a sequence of complex number with limit* $\zeta\in\mathbb{C}$ *and* $\{Q_n\}_{n=m+1}^{\infty}$ *the sequence of monic orthogonal polynomials with respect to the pair* $(\mathcal{L}^{(\alpha,\beta)},\mu)$ *such that* $Q_n(\zeta_n)=0$*, then:*

1. *For every* $d > 1$ *there is a positive number* N_d*, such that* $\{z \in \mathbb{C} : Q_n(z) = 0\} \subset \{z \in \mathbb{C} : |z| \leq \Delta(\zeta) + d\}$ *whenever* $n > N_d$.

2. *If* $\delta(\zeta) > 2$*, the zeros of* Q_n *can not accumulate on* $[-1, 1]$ *and for* n *sufficiently large they are simple.*

Where for $z \in \mathbb{C}$, $\Delta(z) = \sup\limits_{x\in[-1,1]} |z - x|$ *and* $\delta(z) = \inf\limits_{x\in[-1,1]} |z - x|$.

Proof. We already know that $Q_n(\zeta_n) = 0$ and if $Q_n(z) = 0$ then $\widehat{Q}_n(z) = \widehat{Q}_n(\zeta_n)$. From the Gauss–Lucas theorem (cf. [157, §2.1.3]), it is known that the critical points of $\widehat{Q}_n$ lie in the convex hull of its zeros and from 2. of Corollary 2.2 the zeros of the polynomials $\{\widehat{Q}_n\}_{n=0}^{\infty}$ accumulate on $[-1, 1]$. Hence from the *bisector theorem* (see the proof of Theorem 2.6 or [157, §5.5.7]) $|z| \leq \Delta(\zeta_n) + 1$ and the first part of the theorem is established.

To verify the second assertion of the theorem, note that if z is a zero of Q_n, from (2.19) we get

$$\prod_{k=1}^{n} \left| \frac{z - \widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}} \right| = 1. \tag{2.41}$$

Let $\mathcal{V}_\varepsilon([-1, 1]) = \{z \in \mathbb{C} : \delta(z) < \epsilon\}$ be an ε–neighborhood of $[-1, 1]$. On the other hand, as $\lim\limits_{n\to\infty} \zeta_n = \zeta$, then for all $\varepsilon > 0$ there is a $N_\varepsilon > 0$ such that $|\delta(\zeta_n) - \delta(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$.

If $\delta(\zeta) > 2$, let us choose $\varepsilon = \varepsilon_\zeta = \dfrac{1}{2}\,(\delta(\zeta) - 2)$ and suppose that there is a $z_0 \in \mathcal{V}_{\varepsilon_\zeta}([-1, 1])$ such that $Q_n(z_0) = 0$ for some $n > N_{\varepsilon_\zeta}$. Hence

$$\prod_{k=1}^{n} \left| \frac{z_0 - \widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}} \right| < \left(\frac{2 + \varepsilon_\zeta}{\delta(\zeta_n)} \right)^n < 1, \tag{2.42}$$

which is in contradiction with (2.41). Hence $\{z \in \mathbb{C} : Q_n(z) = 0\} \bigcap \mathcal{V}_{\varepsilon_n}([-1, 1]) = \varnothing$ for all $n > N_{\varepsilon_\zeta}$, i.e. the zeros of Q_n can not accumulate on $\mathcal{V}_{\varepsilon_\zeta}([-1, 1])$.

From (2.19) it is straightforward that a multiple zero of Q_n is also a critical point of $\widehat{Q}_n$. But, from 2. of Corollary 2.2 and the Gauss–Lucas theorem the critical point of $\widehat{Q}_n$ accumulate on $[-1, 1]$. Thus, we have that for n sufficiently large the zeros of Q_n are simple.

□

Proof of Theorem 2.1. From (2.19) the zeros of Q_n satisfy the equation

$$\left|\widehat{Q}_n(z)\right|^{\frac{1}{n}} = \left|\widehat{Q}_n(\zeta_n)\right|^{\frac{1}{n}} . \tag{2.43}$$

If $z \in \mathbb{C} \setminus [-1, 1]$, by taking limit when $n \to \infty$, from 1. of Lemma 2.4, and using (2.40) on both sides of (2.43), we have that the zeros of the sequence of polynomials $\{Q_n\}_{n=m}^{\infty}$ cannot accumulate outside the set

$$\left\{ z \in \mathbb{C} : |z + \sqrt{z^2 - 1}| = e^{\eta_\zeta} \right\} \bigcup [-1, 1].$$

Hence $z + \sqrt{z^2 - 1} = e^{\eta_\zeta + i\theta}$ and $z - \sqrt{z^2 - 1} = e^{-(\eta_\zeta + i\theta)}$ for $0 \leq \theta < 2\pi$, where we have that $2z = e^{\eta_\zeta + i\theta} + e^{-(\eta_\zeta + i\theta)}$.

The assertion for $\delta(\zeta) > 2$ is straightforward from 2. of Lemma 2.4.

□

Now, we will state the relative asymptotic between the polynomials $\{Q_n\}_{n=m+1}^{\infty}$ and the corresponding Jacobi polynomials $P_n^{(\alpha,\beta)}$.

Proof of Theorem 2.2.
1.- Let us prove first that

$$\frac{Q_n(z)}{\widehat{Q}_n(z)} = 1 - \frac{\widehat{Q}_n(\zeta_n)}{\widehat{Q}_n(z)} \underset{n}{\rightrightarrows} 1, \tag{2.44}$$

uniformly on compact subsets K of the set $\{z \in \mathbb{C} : |\varphi(z)| > |\varphi(\zeta)|\}$. In order to prove (2.44) it is sufficient to show that

$$\frac{\widehat{Q}_n(\zeta_n)}{\widehat{Q}_n(z)} \underset{n}{\rightrightarrows} 0, \quad \text{uniformly on } K. \tag{2.45}$$

From [163, (8.21.9) and (4.21.6)], we have the well known strong or power asymptotic of the monic Jacobi polynomials

$$\frac{2^n\, P_n^{(\alpha,\beta)}(z)}{\varphi^n(z)} \underset{n}{\rightrightarrows} \left(\frac{\varphi(z)-1}{2(z-1)}\right)^{\alpha} \left(\frac{\varphi(z)+1}{2(z+1)}\right)^{\beta} \sqrt{\frac{\varphi'(z)}{2}}, \tag{2.46}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$. Note that

$$\frac{\widehat{Q}_n(\zeta_n)}{\widehat{Q}_n(z)} = \frac{\widehat{Q}_n(\zeta_n)}{P_n^{(\alpha,\beta)}(\zeta_n)} \frac{P_n^{(\alpha,\beta)}(z)}{\widehat{Q}_n(z)} \frac{2^n\, P_n^{(\alpha,\beta)}(\zeta_n)}{\varphi^n(\zeta_n)} \frac{\varphi^n(z)}{2^n\, P_n^{(\alpha,\beta)}(z)} \left(\frac{\varphi(\zeta_n)}{\varphi(z)}\right)^n.$$

From (2.35) and (2.46) the first four factors in the right hand side of the previous formula have finite limits; meanwhile, the last factor tends to 0 when $n \to \infty$, and we get (2.45). Finally the assertion 1 is straightforward from (2.35).

2.- For the Assertion 2 of the theorem it is sufficient to prove that

$$\frac{Q_n(z)}{\widehat{Q}_n(\zeta_n)} = \frac{\widehat{Q}_n(z)}{\widehat{Q}_n(\zeta_n)} - 1 \underset{n}{\rightrightarrows} -1, \tag{2.47}$$

uniformly on compact subsets K of the set $\{z \in \mathbb{C} : |\varphi(z)| < |\varphi(\zeta)|\} \setminus [-1, 1]$. Note that

$$\frac{\widehat{Q}_n(z)}{\widehat{Q}_n(\zeta_n)} = \frac{\widehat{Q}_n(z)}{P_n^{(\alpha,\beta)}(z)} \frac{P_n^{(\alpha,\beta)}(\zeta_n)}{\widehat{Q}_n(\zeta_n)} \frac{2^n\, P_n^{(\alpha,\beta)}(z)}{\varphi^n(z)} \frac{\varphi^n(\zeta_n)}{2^n\, P_n^{(\alpha,\beta)}(\zeta_n)} \left(\frac{\varphi(z)}{\varphi(\zeta_n)}\right)^n.$$

Now, the first part of the Assertion 2 is straightforward from (2.35).

If $\delta(\zeta) > 2$, let $\mathcal{V}_\varepsilon([-1, 1]) = \{z \in \mathbb{C} : \delta(z) < \epsilon\}$ be a ε–neighborhood of $[-1, 1]$, where $\varepsilon = \varepsilon_\zeta = \dfrac{\delta(\zeta)}{2} - 1$. By the same reasoning that was deduced (2.42) we get that

$$\left|\frac{\widehat{Q}_n(z)}{\widehat{Q}_n(\zeta_n)}\right| < \kappa^n, \quad \text{for all } z \in \mathcal{V}_\varepsilon([-1, 1]), \kappa < 1. \tag{2.48}$$

Hence from the first part of the Assertion 2 and (2.48) we get the second part of the Assertion 2.

□

2.6 Fluid dynamics model of sources and stagnation points

The fluid dynamic interpretation that we will consider in this section was introduced by H. Pijeira et al in [13]. In that paper the hydrodynamic model was a reinterpretation of the electrostatic model studied by H. Pijeira et al in [12]. The difference between the fluid dynamic model in [13] and the model introduced in the present paper is the complex potential used.

Let us consider a flow of an incompressible fluid in the complex plane, due to a system of $n-1$ source points ($n > 1$) fixed at w_i, $1 \leq i \leq n-1$, with unitary rate of fluid emission per unit time (*strength of the source*), and two additional source points at 1 and -1 with strength $a > 0$ and $b > 0$ respectively. Here, a *source* is a point in which the fluid is continuously created and uniformly distributed in all directions with constant strength (*steady source*). Let us call *flow field generated by a Jacobi set of sources* to a flow of a fluid under the above conditions, or simple a *flow field*.

The complex potential of a flow field at any point z (cf. [59, Ch. 10] and [122, Vol. II–Ch. 6]), by the superposition principle of solutions, is given by

$$\begin{aligned}\Upsilon(z) &= \sum_{i=1}^{n-1} \log(z-w_i) + a\,\log(z-1) + b\,\log(z+1),\\ &= \log\left((z-1)^a\,(z+1)^b \prod_{i=1}^{n-1}(z-w_i)\right). \end{aligned} \tag{2.49}$$

From a complex potential Υ, a *complex velocity* $\mathcal{V}$ can be derived by differentiation ($\mathcal{V}(z) = \frac{d\Upsilon}{dz}(z)$). A standard problem associated with the complex velocity is to find the zeros, that correspond to the set of *stagnation points*, i.e. points where the fluid has zero velocity.

We are interested in an inverse problem in the following sense, build a flow field such that the stagnation points are at preassigned points with *nice* properties. As it is well known, the zeros of orthogonal polynomials with respect to a finite positive Borel measure on $[-1,1]$ have a rich set of *nice* properties ([163, Chapter VI]), and will be taken as preassigned stagnation points. Here, we consider that $\mu \in \mathcal{P}_1(\alpha,\beta)$. In the next paragraph the statement of the problem will be established.

Problem. *Let $\{x_1, x_2, \dots, x_n\}$ be the set of zeros of the nth orthogonal polynomial L_n with respect to $\mu \in \mathcal{P}_1(\alpha,\beta)$ with $1 < n$. Build a flow field (location of the source points $w_1, \dots, w_{n-1}$) such that the stagnation points are attained at the points x_i, $i = 1, 2, \dots, n$.*

Let Q_n be a monic polynomial of degree n, whose set of critical points is $\{w_1, w_2, \dots, w_{n-1}\}$, thus

$$\begin{aligned} Q_n'(z) &= n\prod_{i=1}^{n-1}(z-w_i), \quad \Upsilon(z) = \log\left(\frac{1}{n}(z-1)^a\,(z+1)^b\,Q_n'(z)\right),\\ \mathcal{V}(z) &= \frac{\partial \Upsilon}{\partial z}(z) = \frac{\left((z-1)^a\,(z+1)^b\,Q_n'(z)\right)'}{(z-1)^a\,(z+1)^b\,Q_n'(z)} = \frac{\mathcal{L}^{(a-1,b-1)}[Q_n](z)}{(z-1)\,(z+1)\,Q_n'(z)}. \end{aligned}$$

From (2.49) and Theorem 2.6, $\dfrac{\partial \mathcal{V}}{\partial z}(x_k) = 0$ for each stagnation point x_k (zeros of L_n), $k = 1, 2, \dots, n$, i.e.

$$\mathcal{L}^{(a-1,b-1)}[Q_n](x_k) = 0, \quad k = 1,\, 2,\, \cdots, n. \tag{2.50}$$

From Corollary 2.1, there exists a monic polynomial Q_n of degree n, unique up to an additive constant, satisfying equation (2.50), i.e.

$$\mathcal{L}^{(a-1,b-1)}[Q_n](z) = \lambda_n L_n(z), \quad \lambda_n = -n(n+a+b-1). \tag{2.51}$$

Note that (2.51) is the same as (2.6) with $\alpha = a-1$ and $\beta = b-1$. Therefore the $n-1$ source point of the flow field $\{w_1, \dots, w_{n-1}\}$ are the critical point of the nth orthogonal polynomial with respect to the differential operator $\mathcal{L}^{(a-1,b-1)}$.

Answer. *A flow fields generated by a Jacobi set of sources with complex potential* (2.49) *and preassigned stagnation points at the zeros of the nth orthogonal polynomial with respect to the measure $\mu \in \mathcal{P}_1(\alpha,\beta)$ with $n > 1$, has its sources points (with unitary strength) located at the critical points of the nth orthogonal polynomial with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$.*

In Theorem 2.6, we proved that for $m = 1$ all the critical points of Q_n are simple, contained in $[-1, 1]$ and interlace the zeros of L_n. At the beginning of the Section 2.5, we conjectured that this theorem is true for all $m \in \mathbb{N}$. If this were true, then it is not difficult to see that the above model holds for $m \in \mathbb{N}$.

Note that, if we consider a system of electrostatic charges with potential given by (2.49) instead of a system of source points with the same potential function, then we have an analogous electrostatic interpretation.

As it is known, the zeros of the Jacobi polynomials have an electrostatic interpretation (see [163, §6.7]) as the equilibrium points of a certain potential function.

For the case of orthogonality with respect to a differential operator the electrostatic interpretation is an inverse problem in the sense that the equilibrium points are known and the question is to build the electrostatic field.

It would be interesting also the analysis of the stability of the stagnation or equilibrium points, but we shall leave open this problem.

Chapter 3

Orthogonal polynomials with respect to a Laguerre or Hermite operator

3.1 Introduction

In Chapter 2 we have studied analytical and algebraic properties of orthogonal polynomials with respect to a Jacobi differential operator. In Chapter 3 we consider orthogonal polynomials with respect to either a Laguerre or Hermite differential operator and a positive Borel measure measure μ with support contained in $\mathbb{R}$. We want to remark that some of the techniques used in Chapter 2 can not be applied to the asymptotic study of these polynomials. The main difficulty for this case is that we do not have a general result for the relative asymptotic behavior between a sequence of orthogonal polynomials with respect to a measure w supported on the real line or the real semi axis and the sequence of orthogonal polynomials with respect to a positive measure μ, where $d\mu = \frac{1}{\rho}\, dw$ and ρ is polynomial. The most general result known up to date for a relative asymptotic of this kind is for the case in which ρ is a rational function and was given in [101, Th. 3 and Th. 4]. Both theorems require that the rational function ρ satisfies a Lipschitz condition at infinity and that $\rho(\infty) \neq 0$. If ρ is a polynomial of degree m (m even if we have the case of the real line) then ρ do not satisfy the above conditions. Even in the case that we would know the relative asymptotic behavior, it is not clear how to apply this result. Nevertheless, these difficulties can be overcame up to some extent in the study of the normalized sequence of orthogonal polynomials, that is, by scaling the sequence with an appropriate parameter. We study in this chapter some algebraic and analytical properties of the sequence of orthogonal polynomials with respect to a Laguerre or Hermite differential operator. We consider also asymptotic properties of the normalized sequence. We start this introductory chapter by given some definitions and notations.

We denote by $\mathcal{L}_L$ the Laguerre and by $\mathcal{L}_H$ the Hermite differential operators on the space $\mathbb{P}$, i.e.

$$\mathcal{L}_L[f] = xf'' + (1+\alpha - x)f' = x^{-\alpha}\, e^{x}\left(x^{\alpha+1}\, e^{-x}\, f'\right)', \quad f \in \mathbb{P}, \quad \alpha > -1 \tag{3.1}$$

$$\mathcal{L}_H[f] = f'' - 2xf' = e^{x^2}\left(e^{-x^2}\, f'\right)', \quad f \in \mathbb{P}. \tag{3.2}$$

We highlight that $\mathcal{L}_L[f]$ and $\mathcal{L}_H[f]$ are polynomials of the same degree as f, with $f \not\equiv 0$. As is well known, to each one of these second order differential operators we can associate a system of monic polynomials which are both eigenfunctions of the operator and orthogonal with respect to a measure. Let $\{L_n^{\alpha}\}_{n=0}^{\infty}$ be the monic Laguerre polynomials with $\alpha > -1$ and $\{H_n\}_{n=0}^{\infty}$ the monic Hermite polynomials, then

$$\langle L_n^\alpha, L_m^\alpha \rangle_L = \int L_n^\alpha(x) L_m^\alpha(x) dw_L^\alpha(x) \begin{cases} = 0 & \text{if} \quad n \neq m, \\ \neq 0 & \text{if} \quad n = m, \end{cases} \tag{3.3}$$

$$\langle H_n, H_m \rangle_H = \int H_n(x) H_m(x) dw_H(x) \begin{cases} = 0 & \text{if} \quad n \neq m, \\ \neq 0 & \text{if} \quad n = m, \end{cases} \tag{3.4}$$

where $dw_L^\alpha(x) = x^\alpha\, e^{-x} dx, x \in (0, +\infty)$ and $dw_H(x) = e^{-x^2} dx, x \in (-\infty, +\infty)$. In addition,

$$\mathcal{L}_L[L_n^\alpha] = -nL_n \quad \text{and} \quad \mathcal{L}_H[H_n] = -2nH_n. \tag{3.5}$$

To unify the approach, we will denote in the sequel by $\mathcal{L}$ either the Laguerre or Hermite differential operator ($\mathcal{L}_L$ or $\mathcal{L}_H$), by dw the Laguerre or Hermite measure (dw_L^α or dw_H), by L_n the nth Laguerre or Hermite monic orthogonal polynomial (L_n^α or H_n) and by Δ the set $\mathbb{R}_+$ or $\mathbb{R}$, respectively. We will refer to one or the other depending on the case we are solving.

Let μ be a finite positive Borel measure, supported on $\Delta \subset \mathbb{R}$ and $\{P_n\}_{n=0}^\infty$ the corresponding system of monic orthogonal polynomials, i.e.

$$\langle P_n, P_k \rangle_\mu = \int P_n(x) P_k(x) d\mu(x) \begin{cases} \neq 0 & \text{if } n = k, \\ = 0 & \text{if } n \neq k. \end{cases} \tag{3.6}$$

As before, Q_n is the nth monic orthogonal polynomial with respect to the pair $(\mathcal{L}, \mu)$ if $deg[Q_n] = n$ and

$$\int \mathcal{L}[Q_n](x) x^k d\mu(x) = 0 \quad \text{for all} \quad 0 \le k \le n-1, \tag{3.7}$$

or, equivalently,

$$\mathcal{L}[Q_n] = \lambda_n\, P_n, \tag{3.8}$$

where $\lambda_n = -n$ in the Laguerre case and $\lambda_n = -2n$ in the Hermite case.

In the next section we discuss the existence and uniqueness of the orthogonal polynomial with respect to the Laguerre or Hermite differential operator and a suitable measure μ.

The chapter is organized as follows. Section 3.2 is dedicated to the study of existence and uniqueness and on some results concerning the properties of the zeros of orthogonal polynomials with respect to the operators Laguerre or Hermite. In Section 3.3 we show a fluid dynamics model for source point location of a flow of an incompressible fluid with logarithmic velocity–potential in presence of an external field. The study of recurrence relations of these polynomials is done in Section 3.4. Finally, in Section 3.5 we study the asymptotic behavior of the polynomials and their zeros.

3.2 Existence and uniqueness

We are interested in discussing systems of polynomials such that for all $n > m$, for some $m \in \mathbb{N}$, they are solutions of (3.8). Before we prove the existence theorem we prove a preliminary lemma

LEMMA 3.1. *Let n be a fixed natural number and μ a finite positive Borel measure with support contained on $\mathbb{R}$. Then, the differential equation* (3.8) *has a monic polynomial solution Q_n of degree n, which is unique up to an additive constant, if and only if*

$$\int P_n(x) dw(x) = 0. \tag{3.9}$$

where P_n is the nth monic orthogonal polynomials for the measure μ.

Proof. Suppose that there exists a polynomial Q_n of degree n, such that $\mathcal{L}[Q_n] = \lambda_n P_n$, where P_n is the nth monic orthogonal polynomial for μ. From the orthogonality of the sequence $\{L_n\}_{n=0}^{\infty}$ with respect to w,

$$Q_n(z) = L_n(z) + \sum_{k=0}^{n-1} b_{n,k} L_k(z), \tag{3.10}$$

$$P_n(z) = L_n(z) + \sum_{k=0}^{n-1} a_{n,k} L_k(z), \tag{3.11}$$

where $b_{n,k} = \frac{\langle Q_n, L_k\rangle}{\langle L_k, L_k\rangle}$ and $a_{n,k} = \frac{\langle P_n, L_k\rangle}{\langle L_k, L_k\rangle}$.

Replacing Q_n and P_n in (3.8) by the linear combinations (3.10) and (3.11), from the linearity of $\mathcal{L}[\cdot]$ and (3.5) we get

$$a_{n,0} = \frac{\int P_n(x)dw(x)}{\int d\mu} = 0.$$

Conversely, let P_n be the nth monic orthogonal polynomial for μ satisfying (3.9). Let Q_n the polynomial of degree n defined by

$$Q_n(z) = L_n(z) + \sum_{k=0}^{n-1} b_{n,k} L_k(z),$$

where $b_{n,0}$ is an arbitrary constant and

$$b_{n,k} = \frac{\lambda_n}{\lambda_k} a_{n,k} = \frac{\lambda_n}{\lambda_k} \frac{\langle P_n, L_k\rangle}{\langle L_k, L_k\rangle}.$$

Form the linearity of $\mathcal{L}[\cdot]$ and (3.5) we get that $\mathcal{L}[Q_n] = \lambda_n P_n$. □

From the preceding lemma we obtain,

THEOREM 3.1. *Let w be the Laguerre or Hermite measure and μ a finite positive Borel measure on Δ, such that $d\mu(x) = r(x)dw(x)$ with $r \in L^2(w)$. Then, m is the least natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$ if and only if r^{-1} is a polynomial of degree m.*

Proof. Suppose that m is the lowest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. According to Lemma 3.1

$$\int \frac{1}{r(x)} P_n(x)dw(x) = \int P_n(x)dw(x) \begin{cases} = 0 & \text{if } n > m, \\ \neq 0 & \text{if } n = m. \end{cases}$$

But this is equivalent to say that $\dfrac{1}{r(x)} = \sum_{k=0}^{m} c_k L_k(x)$ with $c_m \neq 0$. The converse is straightforward. □

It is possible to give another characterization, in terms of the quasi orthogonality concept, cf. [33], for the existence of a systems of polynomials such that for all $n > m$, for some $m \in \mathbb{N}$, they are solutions of (3.8),

THEOREM 3.2. *Let μ be a finite positive Borel measure on $\mathbb{R}$ and $\{P_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to μ. Then, m is the least natural number such that for each $n > m$ there exists a unique monic polynomial Q_n, except up to an additive constant, orthogonal with respect to the pair $(\mathcal{L}, \mu)$,if and only if for all $n > m$ the polynomial P_n is quasi-orthogonal of index $(m, 1)$ with respect to the measure w.*

Proof. Assume that m is the lowest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. From Lemma 3.1 we have (3.9) holds for $n > m$. From the three term recurrence relation for $\{P_n\}_{n=0}^{\infty}$

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \alpha_n^2 P_{n-1}(x),\ n \geq 1, \\ P_0(x) &= 1,\ P_{-1}(x) = 0,\ \alpha_n, \beta_n \in \mathbb{R} \text{ and } \alpha_n \neq 0, \end{aligned} \tag{3.12}$$

we have that

$$\int P_n(x) x^k dw(x) = 0 \quad \text{for all } \ 0 \leq k < n - m, \tag{3.13}$$

which implies that the polynomial P_n is quasi-orthogonal of index $(n - m, 1)$, cf. [33], with respect to the measure dw (Laguerre or Hermite).

Conversely, assume that m is the lowest natural number such that for $n > m$, the polynomial P_n is quasi-orthogonal of index $(m, 1)$ with respect to the measure dw. We have then that

$$P_n(x) = L_n(x) + \sum_{k=1}^{m} d_{n-k} L_{n-k}(x),$$

which implies that for all integers $n > m$ the polynomials P_n satisfy the condition (3.9). From Lemma 3.1 we have that there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$, for all $n > m$.

□

From the above theorem, we deduce in particular that the differential equation (3.8) has a unique, except up to an additive constant, monic polynomial solution Q_n of degree n *for all the natural numbers* only if $P_n = L_n$ and $d\mu = dw$. Hence $Q_n = L_n$, the polynomial eigenfunctions of $\mathcal{L}$, whose properties are well known.

We define $\mathcal{P}_m[\Delta], m \in \mathbb{N}$, as the class of finite positive measures such that $d\mu(x) = \dfrac{dw(x)}{\rho(x)}$, where ρ is a polynomial of degree m. Also, for $m = 2$ we shall denote $\widetilde{\mathcal{P}}_2[\mathbb{R}]$ the class of measures of the form $d\mu = \dfrac{e^{-x^2}}{x^2 + x_1^2} dx, x_1 \neq 0$ in the Hermite case.

Consider now $\{\zeta_n\}_{n=m+1}^{\infty}$ a sequence of complex numbers, where m is a natural number and assume that $\mu \in \mathcal{P}_m[\Delta]$. We complement the definition of the sequence $\{Q_n\}_{n=m+1}^{\infty}$ in (3.7), considering that henceforth Q_n for each $n > m$ is the polynomial solution of the initial value problem

$$\left\{ \begin{aligned} \mathcal{L}[y] &= \lambda_n P_n, \quad n > m, \\ y(\zeta_n) &= 0, \end{aligned} \right. \tag{3.14}$$

and we say that $\{Q_n\}_{n=m+1}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $Q_n(\zeta_n) = 0$.

Let $\widehat{Q}_n$ be the monic polynomial of degree n ($n > m$) defined by the formula

$$\begin{aligned} \widehat{Q}_n(z) &= \sum_{k=0}^{m} \frac{\lambda_n}{\lambda_{n-k}} b_{n,n-k}\, L_{n-k}(z), \text{ where} \\ b_{n,n-k} &= \frac{1}{\tau_{n-k}} \int P_n(x) L_{n-k}(x) dw(x) \end{aligned} \tag{3.15}$$

and

$$\tau_n = \|L_n\|_w^2 = \int L_n^2(x) dw(x) = \left\{ \begin{array}{rl} n!\ \Gamma(n + \alpha + 1) & \text{Laguerre case,} \\ n! \sqrt{\pi} 2^{-n} & \text{Hermite case.} \end{array} \right. \tag{3.16}$$

Then the initial value problem (3.14) has unique polynomial solution

$$y(z) = Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n) \tag{3.17}$$

3.2.1 The polynomial $\widehat{Q}_n$

Let us start by noting that the polynomials Q_n and $\widehat{Q}_n$ are primitives of the same polynomial Q_n' (or $\widehat{Q}_n'$). From (3.15)

$$\int \widehat{Q}_n(x)\, x^k\, dw(x) = 0, \quad k = 0, 1, \ldots, n-m-1, \tag{3.18}$$

Applying classical arguments [158], it is not difficult to prove the following result, which will be used in the sequel.

PROPOSITION 3.2.1. *The polynomial $\widehat{Q}_n$ defined by (3.15) for all $n > m$, has at least $(n-m)$ zeros and $(n-m-1)$ critical points of odd multiplicity on Δ.*

The following Proposition shows some results concerning the zeros of $\widehat{Q}_n$ and $\widehat{Q}_n'$

PROPOSITION 3.2.2. *Assume that $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$, then the zeros of $\widehat{Q}_n$ and $\widehat{Q}_n'$ are real and simple. The critical points of Q_n interlace the zeros of P_n.*

Proof. Laguerre case. If $m = 1$ and $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ from Proposition 3.2.1 the polynomial $\widehat{Q}_n$ has at least $(n-1)$ real zeros of odd multiplicity on $\mathbb{R}_+$. But, $\widehat{Q}_n$ is a polynomial with real coefficients and degree n, consequently the zeros of $\widehat{Q}_n$ are real and simple. As $Q_n^{'} = \widehat{Q}_n^{'}$, from Rolle's Theorem all the critical points of Q_n are real, simple and $(n-2)$ of them are contained on $\mathbb{R}_+^* =]0, \infty[$.

Denote $G(z) = x^{\alpha+1}\, e^{-x}\, Q_n'(z)$, with $\alpha \in]-1, \infty[$. Notice that G is a real–valued, continuous and differentiable function on $\mathbb{R}_+^*$. Suppose that there exists $x \in \mathbb{R}_+^*$ such that $G(x) = 0$. As $G(0) = 0$ from Rolle's Theorem there exists $x' \in \mathbb{R}_+^*$ such that $G'(x') = 0$. But, $G'(x) = x^\alpha\, e^{-x} \mathcal{L}_L[Q_n] = \lambda_n x^\alpha\, e^{-x}\, P_n(x)$ and all the critical points of G are contained on $\mathbb{R}_+^*$. Hence all the critical points of Q_n belong to $\mathbb{R}_+^*$.

Hermite case. Consider now $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$, that is, $m = 2$ and $d\mu(x) = \dfrac{e^{-x^2}}{x^2 + x_1^2} dx$, $x_1 \neq 0$. Using the relations (3.15) and [163, 5.6.1] we have that for $k > 1$

$$\begin{aligned} \widehat{Q}_{2k}(z^2) &= L_k^{-1/2}(z^2) + \frac{k}{k-1} \frac{\int P_{2k}(x) H_{2k-2}(z) e^{-x^2} dx}{\int H_{2k-2}^2(z) e^{-x^2} dx} L_{k-1}^{-1/2}(z^2) \\ \widehat{Q}_{2k+1}(z^2) &= z L_k^{1/2}(z^2) + \frac{2k+1}{2k-1} \frac{\int P_{2k+1}(x) H_{2k-1}(z) e^{-x^2} dx}{\int H_{2k-1}^2(z) e^{-x^2} dx} z L_{k-1}^{1/2}(z^2) \end{aligned} \tag{3.19}$$

As $L_n^{-1/2}(z^2), z L_n^{1/2}(z^2)$ are the $2n$ and $2n+1$ monic orthogonal polynomials of degree $2n$ and $2n+1$ respectively with respect to the measure $d\mu(x) = e^{-x^2} dx$, from (3.19) and [163, Th. 3.3.4] we have that the zeros of $\widehat{Q}_n, n > 2$ are real.

The statement that critical points of Q_n interlace the zeros of P_n follows by applying Rolle's Theorem to the functions $G(z) = x^{\alpha+1}\, e^{-x}\, Q_n'(z)$ and $G(z) = e^{-x^2}\, Q_n'(z)$, for both the Laguerre and Hermite cases.

□

We conjecture that Proposition 3.2.2 is still valid for any measure in the class $\mathcal{P}_m[\Delta]$, $m > 1$, for the Laguerre case or $m > 2$, m even, for the Hermite case.

3.3 A fluid dynamics model

In this section we show an hydrodynamical model for the zeros of the orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$. As we shall see, for the zeros of the derivatives of these polynomials it is possible to build a similar model.

3.3.1 Hydrodynamical interpretation of the zeros of Q_n'

Here, we consider two fluid dynamics models connected with the type of orthogonality we introduced above. Let a system of $n-1$ points w_i, $1 \leq i \leq n-1$, be given. In the Laguerre model we associate to the system of points the following potential.

$$\mathcal{V}_L(z) := \sum_{i=1}^{n-1} \log \frac{1}{(z-w_i)} - z + (1+\alpha) \log \frac{1}{z}. \tag{3.20}$$

That is, $\mathcal{V}_L$ equals the sum of a *source* with strength equal to unity plus a uniform *stream* 1 at infinity and a source fixed at the origin with constant strength $\alpha + 1$, see [135, Chap. VIII] for the terminology.

For the Hermite model

$$\mathcal{V}_H(z) := \sum_{i=1}^{n-1} \log \frac{1}{(z-w_i)} - z^2. \tag{3.21}$$

A standard problem associated with the velocity–potential is to find the *stagnation points*, i.e. points where the fluid has zero velocity (cf. [59, Chap. 10] and [135, Chap. VIII]).

We are interested in an inverse problem in the following sense, build $n-1$ system (location of the points w_i) such that the stagnation points are at preassigned points with *nice* properties. As is well known, the zeros of orthogonal polynomials with respect to a finite positive Borel measure on $\mathbb{R}$ have a rich set of *nice* properties (cf. in [163, Chapter VI]), and will be taken as preassigned stagnation points. Here we consider that $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$.

Problem. *Let $\{x_1, x_2, \ldots, x_n\}$ be the set of zeros of the nth orthogonal polynomial P_n, ($n > 1$ for the Laguerre case, $n > 2$ for the Hermite case), with respect to a positive Borel measure $\mu \in \mathcal{P}_1[\Delta]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$. Suppose a flow is given, with velocity potential equal to $\mathcal{V}_L$ or $\mathcal{V}_H$. Build a $n-1$ system (location of the source points $w_1, \ldots, w_{n-1}$) such that the stagnation points are attained at the points $z = x_i$, with $i = 1, 2, \ldots, n$.*

First, let us consider the Laguerre case. From (3.20), if $\{x_1, x_2, \ldots, x_n\}$ is the set of stagnation points we have that $\dfrac{\partial \mathcal{V}_L}{\partial z}(x_k) = 0$, that is,

$$\sum_{i=1}^{n-1} \frac{1}{x_k - w_i} - 1 + \frac{1+\alpha}{x_k} = 0, \quad k = 1,\, 2,\, \ldots, n. \tag{3.22}$$

Let R_n be a monic polynomial of degree n with $n-1$ distinct critical points $\{w_1, \ldots, w_{n-1}\} \neq \{x_1, \ldots, x_n\}$, thus $R_n'(x) = n \displaystyle\prod_{i=1}^{n-1}(x - w_i)$. Now, we can rewrite (3.22) as

$$\frac{R_n''(x_k)}{R_n'(x_k)} - 1 + \frac{1+\alpha}{x_k} = 0, \quad k = 1,\, 2,\, \ldots, n$$

or, equivalently,

$$x_k R_n''(x_k) + (1 + \alpha - x_k) R_n'(x_k) = 0, \quad k = 1,\, 2,\, \ldots, n.$$

Therefore, if R_n' has simple zeros, $xR_n''(x) + (1 + \alpha - x)R_n'(x)$ is a polynomial of degree n, with leading coefficient $\lambda_n = -n$, with vanishes at the zeros of P_n, so

$$xR_n''(x) + (1 + \alpha - x)R_n'(x) = \lambda_n P_n(x), \quad \lambda_n = -n. \tag{3.23}$$

Notice that (3.23) is equal to (3.8). From Proposition 3.2.2, the zeros of R_n' are real and simple and $R_n'(x_k) \neq 0$; therefore, $R_n = Q_n$ and the $n-1$ source points of the field are the critical points of Q_n (or $\widehat{Q}_n$).

In a similar way, for the Hermite case, from (3.21), if $\{x_1, x_2, \ldots, x_n\}$ is the set of stagnation points then $\dfrac{\partial \mathcal{V}_H}{\partial z}(x_k) = 0$, and

$$\sum_{i=1}^{n-1} \frac{1}{x_k - w_i} - 2x_k = 0, \quad k = 1,\, 2,\, \cdots, n.$$

Let R_n be a monic polynomial of degree n with $n-1$ distinct critical points $\{w_1, \ldots, w_{n-1}\} \neq \{x_1, \ldots, x_n\}$, thus $R_n'(x) = n\prod_{i=1}^{n-1}(x - w_i)$, which implies that

$$R_n''(x_k) - 2x_k R_n'(x_k) = 0 \quad k = 1,\, 2,\, \cdots, n.$$

Consequently $R_n''(x) - 2xR_n'(x)$ is a polynomial of degree n, with leading coefficient $\lambda_n = -2n$, that vanishes at the zeros of P_n. Therefore

$$R_n''(x) - 2xR_n'(x) = \lambda_n P_n(x), \quad \lambda_n = -2n,$$

which is equal to (3.8). From proposition 3.2.2, the zeros of R_n' are simple and $R_n'(x_k) \neq 0$, Therefore, $R_n = Q_n$ and the $n-1$ source points of the field are the critical points of Q_n (or $\widehat{Q}_n$). As a conclusion, we get

> **Answer.** *The flow of an incompressible two–dimensional fluid, due to a $(n-1)$ system located at the critical points of the n-th orthogonal polynomial with respect to $(\mathcal{L}, \mu)$ with $\mu \in \mathcal{P}_1[\Delta]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$ and velocity potential $\mathcal{V}_L$ or $\mathcal{V}_H$ has its $n-1$ stagnation points at the $n-1$ critical points of Q_n.*

3.3.2 Hydrodynamical interpretation of the zeros of $\widehat{Q}_n$

Let us consider a flow of an incompressible fluid in the complex plane, due to a system of n *different* points $(n > 1)$ fixed at w_i, $1 \leq i \leq n$. At each point w_i of the system there is defined a complex potential $\mathcal{V}_i$ which for the Laguerre case equals to the sum of a *source(sink)* with a fixed strength $\mathrm{Re}[c_i]$ plus a *vortex* with a fixed strength $\Im[c_i]$ plus a *uniform stream* U_i at infinite. Here c_i, d_i are fixed complex number which depends of the the position of the set of remaining points $\{w_i\}_{i=1}^n$, see [135, pag 200] for the terminology. We shall call n *system* to the set of the n points fixed at w_i with its respective potential of velocities.

Define the functions $f_i(w_1, \ldots, w_n) = \dfrac{R_n''(w_i)}{R_n'(w_i)}, i = 1, \ldots, n$ where $R_n(z) = \prod_{i=1}^{n}(z - w_i)$. The complex potential at any point z, due to the above system, for the Laguerre case is

$$\begin{aligned}\mathcal{V}_L(z; w_1, \ldots, w_n) := \sum_{i=1}^{n} \mathcal{V}_{L,i} = \sum_{i=1}^{n} z(f_i(w_1, \ldots, w_n) - 1) &+ \\ (1 + \alpha + w_i(f_i(w_1, \ldots, w_n) - 1)) \log(z - w_i)&,\end{aligned} \tag{3.24}$$

for the Hermite case will be

$$\mathcal{V}_H(z; w_1, \ldots, w_n) := \sum_{i=1}^{n} \mathcal{V}_{H,i} = \sum_{i=1}^{n} -2z + (f_i(w_1, \ldots, w_n) - 2w_i) \log(z - w_i), \tag{3.25}$$

As in the preceding section, we are interested in the problem. Build a n system(location of the points $w_1, \ldots, w_n$) such that the stagnation points are on preassigned points with *nice* properties. As is well known, the zeros of orthogonal polynomials with respect to a finite positive Borel measures on $\mathbb{R}$ have a rich set of *nice* properties (cf. in [163, Chapter VI]), and will be taken as preassigned stagnation points. Here we consider $\mu \in \mathcal{P}_1[\Delta]$ or or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$. In the next paragraph will establish the statement of the problem for both Laguerre and Hermite cases.

> **Problem.** *Let $\{x_1, x_2, \ldots, x_n\}$ be the set of zeros of the nth orthogonal polynomial P_n, ($n > 1$ for the Laguerre case, $n > 2$ for the Hermite case) with respect to a positive Borel measure $\mu \in \mathcal{P}_1[\Delta]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$. Suppose given a flow, with complex potential for the Laguerre case equal to $\mathcal{V}_L$ and for the Hermite case equal to $\mathcal{V}_H$. Build a n system (location of the points $w_1, \ldots, w_n$) such that the stagnation points are attained at the points $z = x_i$, with $i = 1, 2, \ldots, n$.*

Consider firstly the Laguerre case. Let us rewrite (3.24) as

$$\mathcal{V}_L(z; w_1, \ldots, w_n) = \sum_{i=1}^{n} -z + (1 + \alpha - w_i) \log(z - w_i) + \sum_{i=1}^{n} z f_i(w_1, \ldots, w_n) + w_i f_i(w_1, \ldots, w_n) \log(z - w_i).$$

If x_k is a stagnation point then $\dfrac{\partial \mathcal{V}_L}{\partial z}(x_k) = 0$, this gives

$$(1 + \alpha - x_k) \sum_{i=1}^{n} \frac{1}{x_k - w_i} + x_k \sum_{i=1}^{n} \frac{R_n''(w_i)}{R_n'(w_i)(x_k - w_i)} = 0, \quad k = 1, 2, \ldots, n. \tag{3.26}$$

We are looking for a solution $R_n(z) = \prod_{i=1}^{n}(z - w_i)$, with $w_i \neq w_j \neq x_k, \forall i, j, k$ such that (3.26) holds. This assumption implies that the sum in the second term of the left hand side of expression (3.26) equals to

$$\sum_{i=1}^{n} \frac{R_n''(w_i)}{R_n'(w_i)(x_k - w_i)} = \frac{R_n''(x_k)}{R_n(x_k)}, \quad k = 1, 2, \ldots, n.$$

Therefore, (3.26) is equivalent to

$$(1 + \alpha - x_k) R_n'(x_k) + x_k R_n''(x_k) = 0, \quad k = 1, 2, \ldots, n.$$

Note that $x R_n''(x) + (1 + \alpha - x) R_n'(x)$ is a polynomial of degree n, with leading coefficient $\lambda_n = -n$ and that vanishes at the zeros of P_n, i.e.

$$x R_n''(x) + (1 + \alpha - x) R_n'(x) = \lambda_n P_n(x), \quad \lambda_n = -n. \tag{3.27}$$

Observe that expression (3.27) is equivalent to (3.8). From Proposition 3.2.2, the zeros of $\widehat{Q}_n, \widehat{Q}_n'$ are real and simple and $Q_n'(x_k) \neq 0$, therefore, $R_n = \widehat{Q}_n$ is a solution. Hence, a solution to our problem yields the n points as the n zeros of the polynomial $\widehat{Q}_n$.

For the Hermite case we have a similar situation. Note that the potential in (3.25) can be written equivalently as

$$-2\sum_{i=1}^{n} z + w_i \log(z - w_i) + \sum_{i=1}^{n} f_i(w_1, \ldots, w_n) \log(z - w_i).$$

Hence, if x_k is a stagnation point then $\dfrac{\partial \mathcal{V}_H}{\partial z}(x_k) = 0$ this gives

$$2x_k \sum_{i=1}^{n} \frac{1}{x_k - w_i} - \sum_{i=1}^{n} \frac{R_n''(w_i)}{R_n'(w_i)(x_k - w_i)} = 0, \quad k = 1, 2, \ldots, n. \tag{3.28}$$

Again, we can deduce that expression (3.28) equals to

$$R_n''(x_k) - 2x_k R_n'(x_k) = 0, \quad k = 1, 2, \ldots, n. \tag{3.29}$$

Note that $xR_n''(x) + (1 + \alpha - x)R_n'(x)$ is a polynomials of degree n, with leading coefficient $\lambda_n = -n$ and that vanishes at the zeros of P_n, i.e.

$$xR_n''(x) + (1 + \alpha - x)R_n'(x) = \lambda_n P_n(x), \quad \lambda_n = -n. \tag{3.30}$$

Therefore, expression (3.30) is equivalent to (3.8) which implies $R_n = \widehat{Q}_n$ is a solution to our problem. As a conclusion,

Answer. *The flow of an incompressible two–dimensional fluid, due to n points with complex potential $\mathcal{V}_L$ given by (3.24) for the Laguerre case and $(\mathcal{V}_H)$ given by (3.25) for the Hermite case, located in the zeros of the n-th orthogonal polynomial $\widehat{Q}_n$ with respect to $(\mathcal{L}, \mu)$, with $\mu \in \mathcal{P}_1[\Delta]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$ has its n stagnation points in the n zeros of the nth orthogonal polynomial $\widehat{Q}_n$.*

Notice that if we consider electrostatic charges instead of source points we have an analogous electrostatic model. It would be interesting to consider the uniqueness of the solution obtained, in other words, what can be said about the solutions of the form $Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n)$ and to extend this model to more general classes of measures μ. It would also of interest to decide if these stagnation or equilibrium points are stable, but we shall leave open all these problems.

3.4 Recurrence relations

Notice the fundamental role of the polynomials $\widehat{Q}_n, Q_n'$ and their zeros in the fluid dynamic models studied in the previous section. So, our goal here is to obtain a recurrence formula to compute these polynomials.

Let $m \in \mathbb{N}$ be a fixed number, $\{\zeta_n\}_{n=0}^{\infty}$ a sequence of complex numbers, and $\mu \in \mathcal{P}_m[\Delta]$, then for all $n > m$ the polynomials Q_n (solution of (3.14)) are uniquely determined by (3.15)–(3.17). Without loss of generality, we will complete the sequence of polynomials Q_n for all $n \in \mathbb{N}$ as

$$Q_n(z) = (L_n(z) - L_n(\zeta_n)) + \lambda_n \sum_{k=1}^{\min(m,n)} \frac{b_{n,n-k}}{\lambda_{n-k}} \left(L_{n-k}(z) - L_{n-k}(\zeta_n)\right). \tag{3.31}$$

Notice that $\{Q_n\}$ defined by (3.31) is a sequence of polynomials, such that $Q_n(\zeta_n) = 0$ for all $n \geq 1$. Let us remark that if $n \leq m$, in general, $\mathcal{L}[Q_n] \neq \lambda_n P_n$.

Additionally, as the degree of a polynomial is invariant by evaluation in $\mathcal{L}[\cdot]$ and the polynomial Q_n, for all $n \leq m$, is of degree n, the next sequence of polynomials is a system of polynomials too

$$\{1, \mathcal{L}[Q_1], \ldots, \mathcal{L}[Q_m], P_{m+1}, \ldots, P_n, \ldots\}.$$

Recurrence relation for $\{Q_n'\}$

The objective of this subsection is to prove

THEOREM 3.3. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m[\Delta]$. Then if R is any primitive of ρ, for each $n > (2m+1)$ the sequence of polynomials Q_n' satisfy the relation*

$$R(z)Q_n'(z) = \sum_{k=-m-1}^{m+1} \beta_{(R,n,n-k)}\, Q_{n-k}'(z), \tag{3.32}$$

where the initial values $Q_{m+1}', \ldots, Q_{2m+2}'$ are given by the derivatives of (3.15) and

$$\begin{aligned} \beta_{R,n,n-k} &= \frac{1}{\lambda_{n-k}} (\lambda_n\, e_{R,n,n-k} + d_{n,n-k}), \\ e_{R,n,n-k} &= \frac{1}{l_{n-k}} \langle RP_n, P_{n-k}\rangle_\mu. \end{aligned}$$

and $d_{n,n-k}$ is defined for the Laguerre and Hermite cases as

$$d_{n,n-k}^L = \frac{1}{l_{n-k}} \sum_{j=0}^{m+1} \dot{b}_{n,n-j}\, b_{n-k,n-j}, \tag{3.33}$$

$$d_{n,n-k}^H = \frac{n}{l_{n-k}} \sum_{j=0}^{m+1} \tau_{n-j}\, b_{(n,n+1-j)}\, b_{n-k,n-j}, \tag{3.34}$$

$$l_i = \langle P_i, P_i\rangle_\mu, \quad \tau_i = \langle L_i, L_i\rangle_w,$$

$$\dot{b}_{n,n-k} = \begin{cases} n\tau_n & \text{if} \quad k = 0, \\ n\tau_{n-k}\left(b_{n,n-k} - (n+1-k-\alpha) b_{(n,n+1-k)}\right) & \text{if} \quad 1 \le k \le m, \\ -n\tau_{n-m}(n-m-\alpha)\, b_{(n,n-m)} & \text{if} \quad k = m+1. \end{cases}$$

First, we prove the following preliminary lemma

LEMMA 3.2. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m[\Delta]$. Then for $n > m$, the polynomials Q_n' satisfy the following relations:*

$$z\rho(z)Q_n'(z) = \sum_{k=-m}^{m+1} d_{n,n-k}^L\, P_{n-k}(z), \quad \textit{Laguerre case} \tag{3.35}$$

$$\rho(z)Q_n'(z) = \sum_{k=-m}^{m+1} d_{n,n-k}^H\, P_{n-k}(z), \quad \textit{Hermite case} \tag{3.36}$$

where $d_{n,n-k}^L, d_{n,n-k}^L$ as in (3.33),(3.34)

Proof. For the Laguerre case, from (3.5), (3.15) and the structure relation

$$z\, L_{n-k}'(z) = (n-k)\, \left(L_{n-k}(z) - (n-k-\alpha)\, L_{n-k-1}(z)\right),$$

satisfied by the monic Laguerre polynomials (cf. [163, (5.1.8),(5.1.14)]), we have

$$\begin{aligned} z\widehat{Q}'_n(z) &= \sum_{k=0}^{m} a_{n,n-k}\; zL'_{n-k}(z) \\ &= n\sum_{k=0}^{m} b_{n,n-k}L_{n-k}(z) - n\sum_{k=1}^{m+1}(n+1-k-\alpha)b_{(n,n+1-k)}\; L_{n-k}(z) \\ &= \sum_{k=0}^{m+1} \frac{\dot{b}_{n,n-k}}{\tau_{n-k}}\; L_{n-k}(z). \end{aligned}$$

Hence

$$\begin{aligned} \int x\widehat{Q}'_n(x)\rho(x)P_\nu(x)d\mu(x) &= \int x\widehat{Q}'_n(x)P_\nu(x)dw(x) \\ &= \sum_{k=0}^{m+1} \frac{\dot{b}_{n,n-k}}{\tau_{n-k}} \int L_{n-k}(z)P_\nu(x)dw(x) \\ &= \sum_{k=0}^{m+1} \dot{b}_{n,n-k}\; b_{(\nu,n-k)}. \end{aligned}$$

As $b_{(\nu,n-k)} = 0$ for $\nu < n-k$ and $0 \le k \le m+1$, we have that $b_{(\nu,n-k)} = 0$ for $\nu = 0,\ldots,n-m-2$ and (3.35).

For the case Hermite, using the corresponding structure relation for monic Hermite polynomials [163, (5.5.10),(5.5.6)] (i.e. $H'_n = nH_{n-1}$) and following the same method as above we have (3.36). □

We have then

Proof. (*of Theorem 3.3*)

The polynomial RQ'_n can be expanded as a linear combination of the polynomials $\{Q'_n\}_{n=1}^{\infty}$, thus, there exist $(n+m)$ constants $\beta_{(R,n,1)},\ldots,\beta_{(R,n,n+m)}$ such that

$$R(z)Q'_n(z) = \sum_{k=-m}^{n-1} \beta_{R,n,n-k}\, Q'_{n-k}(z).$$

• Laguerre case. Let $\Omega_\alpha(z) = z\left(R(z)Q'_n(z)\right)' + (1+\alpha-z)R(z)Q'_n(z)$. Then, on the one hand,

$$\begin{aligned} \Omega_\alpha(z) &= \sum_{k=-m}^{n-1} \beta_{R,n,n-k}\, \mathcal{L}_L\left[Q_{n-k}\right](z) \\ &= \sum_{k=-m-1}^{n-m-1} \beta_{R,n,n-k}\, \lambda_{n-k}\, P_{n-k}(z) + \sum_{k=n-m}^{n-1} \beta_{R,n,n-k}\, \mathcal{L}_L\left[Q_{n-k}\right](z). \end{aligned} \tag{3.37}$$

On the other hand

$$\Omega_\alpha(z) = R(z)\mathcal{L}_L[Q_n](z) + z\rho(z)Q'_n(z) = \lambda_n R(z)\, P_n(z) + z\rho(z)Q'_n(z). \tag{3.38}$$

From (3.6)

$$R(z)\, P_n(z) = \sum_{k=-m-1}^{m+1} e_{R,n,n-k}\, P_{n-k}(z). \tag{3.39}$$

Substituting (3.35) and (3.39) in (3.38), we obtain

$$\Omega_\alpha(z) = \sum_{k=-m-1}^{m+1} \left(\lambda_n \, e_{R,n,n-k} + d^L_{n,n-k}\right) \, P_{n-k}(z). \tag{3.40}$$

As $n \geq 2(m+1)$, we can assume that (3.40) is the expansion of Ω_α in terms of the previous mixed system of polynomials. Now, comparing coefficient between (3.37) and (3.40) we have that $\beta_{R,n,n-k} = 0$ for all $k = 1, \ldots, n-m-2$, and we have the relation (3.32).

• Hermite case. The proof is analogous to the Laguerre case, with $\Omega(z) = (R(z)Q_n'(z))' - 2zR(z)Q_n'(z)$ and (3.36), instead of Ω_α and (3.35). □

Recurrence relation for $\{\widehat{Q}_n\}$

In the following result, we show that for $n > n_0$, for some n_0, the system of polynomials $\{Q_n\}_{n=3m+2}^{\infty}$ satisfy a recurrence relation with a fixed finite number of terms.

THEOREM 3.4. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m[\Delta]$, for each $n > 3m+1$ the sequence $\{\widehat{Q}_n\}$ satisfies that*

$$H(z)\widehat{Q}_n(z) = \sum_{k=-2m-1}^{2m+1} \vartheta_{n,n-k}\widehat{Q}_{n-k}(z), \tag{3.41}$$

where where the initial values $\widehat{Q}_{m+1}, \ldots, \widehat{Q}_{2m+2}$ are given by (3.15) *and H is any primitive of the function ρ^2, R is any primitive of the function ρ and*

$$\begin{aligned}
\vartheta_{n,n-k} &= \frac{1}{\lambda_{n-k}}\left(\lambda_n e_{H,n,n-k} + \widetilde{e}_{\rho,n,n-k} + \widetilde{e}_{h_{m+1},n,n-k}\right), \\
e_{H,n,k} &= \left(\int P_n(z)\, P_{n-k}(x)\, H(x)\, d\mu(x)\right)\left(\int P^2_{n-k}(x) d\mu(x)\right)^{-1}, \\
\widetilde{e}_{\rho,n,n-k} &= \sum_{j=\max\{-m,k-m-1\}}^{\min\{m,k+m+1\}} d_{k-j,n-k+j}\, e_{\rho,n-k+j,n-k}, \\
\widetilde{e}_{h_{m+1},n,n-k} &= \sum_{j=\max\{-m,k-m-1\}}^{\min\{m,k+m+1} \widehat{b}_{n,n-j}\, e_{h_{m+1},n-j,n-k}. \\
\widehat{b}_{n,n+k} &= \frac{\lambda_n}{l_{n+k}} \sum_{j=\max\{0,-k\}}^{\min\{m,k-m\}} \frac{\tau_j}{\lambda_j}\, b_{n,j} b_{n+k,j},
\end{aligned}$$

and $d_{n,n-k}$ as in (3.33),(3.34).

We shall prove some previous lemmas.

LEMMA 3.3. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m[\Delta]$. Then for $n > m$*

$$P_n(z) = \sum_{k=0}^{m} b_{n,n-k}\, L_{n-k}(z), \tag{3.42}$$

$$\rho(z)L_n(z) = \tau_n \sum_{k=0}^{m} \frac{b_{(n+k,n)}}{l_{n+k}}\, P_{n+k}(z) \tag{3.43}$$

$$T(z)\, P_n(z) = \sum_{k=-k}^{k} e_{T,n,n-k}\, P_{n-k}(z), \tag{3.44}$$

$$\sum_{j=-m-1}^{m+1} 2d_{n,n-j}\, \rho(z)\, P_{n-j}(z) = \sum_{k=-2m-1}^{2m+1} \widetilde{e}_{\rho,n,n-k}\, P_{n-k}(z) \tag{3.45}$$

where T is an arbitrary polynomial of degree k and

$$\begin{aligned}
b_{(i,j)} &= \frac{1}{\tau_j} \left\langle P_i, L_j \right\rangle_w, \\
\tau_j &= \left\langle L_j, L_j \right\rangle_w, \\
&= n!\Gamma(\alpha+1)\binom{n+\alpha}{n} \quad \textit{Laguerre case,} \\
&= \pi^{1/2} 2^{-n} n! \quad \textit{Hermite case,} \\
l_i &= \int P_i^2(x) d\mu(x), \\
e_{T,n,k} &= \left(\int P_n(z)\, P_{n-k}(x)\, T(x)\, d\mu(x) \right) \left(\int P_{n-k}^2(x) d\mu(x) \right)^{-1}, \\
\widetilde{e}_{\rho,n,n-k} &= \sum_{j=\max\{-m,k-m-1\}}^{\min\{m,k+m+1\}} 2d_{k-j,n-k+j} e_{\rho,n-k+j,n-k}.
\end{aligned}$$

Proof. Relation (3.42) is straightforward from the Fourier expansion of P_n in terms of the polynomials $\{L_k\}$, $k = 0, 1, \ldots, n$.

From orthogonality relations and the relation between μ and w, if $i > j + m$

$$\int L_i(x) L_j(x) dw(x) = \int L_i(x) L_j(x)\, \rho(x)\, d\mu(x) = 0. \tag{3.46}$$

The relation (3.43) is straightforward from the Fourier expansion of the function $\rho\, L_n$ in terms of the basis $\{P_k\}$, $k = 0, 1, \ldots, n+m$ and (3.46).

Relation (3.44) is immediate from the definition of orthogonality of P_n. Relation (3.45) follows from

$$
\begin{aligned}
\sum_{j=-m-1}^{m+1} 2d_{n,n-j}\ \rho(z)\,P_{n-j}(z) &= \sum_{j=-m-1}^{m+1} 2d_{n,n-j} \sum_{k=-m}^{m} e_{\rho,n-j,n-j-k}\,P_{n-j-k}(z)\\
&= \sum_{j=-m}^{m} 2d_{k-j,n-k+j} \sum_{k=-j-m-1}^{j+m+1} e_{\rho,n-k+j,n-k}\,P_{n-k}(z)\\
&= \sum_{k=-2m-1}^{2m+1} \left(\sum_{j=\max\{-m,k-m-1\}}^{\min\{m,k+m+1\}} 2d_{k-j,n-k+j} e_{\rho,n-k+j,n-k} \right) P_{n-k}(z)\\
&= \sum_{k=-2m-1}^{2m+1} \widetilde{e}_{\rho,n,n-k}\,P_{n-k}(z).
\end{aligned}
$$

□

LEMMA 3.4. *The following relations hold*

$$
\begin{aligned}
\rho(z)\widehat{Q}_n(z) &= \sum_{k=-m}^{m} \widehat{b}_{n,n+k}\,P_{n+k}(z), &(3.47)\\
h_{m+1}(z)\rho(z)\widehat{Q}_n(z) &= \sum_{k=-2m-1}^{2m+1} e_{h_{m+1},n,n-k}\,P_{n-k}(z), &(3.48)\\
z\rho(z)\widehat{Q}'_n(z) &= \sum_{k=-m-1}^{m+1} d_{n,n-k}\ P_{n-k}(z) \quad \textit{Laguerre case}, &(3.49)\\
\rho(z)\widehat{Q}'_n(z) &= \sum_{k=-m-1}^{m+1} d_{n,n-k}\ P_{n-k}(z), \quad \textit{Hermite case}, &(3.50)
\end{aligned}
$$

where h_{m+1} is any polynomial of degree $m+1$ and

$$
\begin{aligned}
\widehat{b}_{n,n+k} &= \frac{\lambda_n}{l_{n+k}} \sum_{j=j_1}^{j_2} \frac{\tau_j}{\lambda_j}\, b_{n,j} b_{n+k,j},\\
&\quad j_1 = \max\{0,-k\} \text{ and } j_2 = \min\{m, k-m\}.\\
d_{n-k,k} &= \frac{1}{l_{n-k}} \sum_{j=j_3(k)}^{j_4(k)} \tau_{n-j}\, \widetilde{c}_{n-j,j}\, b_{n-k,n-j},\\
&\quad j_3(k) = \max\{1,k\} \text{ and } j_4(k) = \min\{m+1, m+k\},\\
\widetilde{c}_{n-k,k} &= \lambda_n \sum_{j=j_5(k)}^{j_6(k)} \frac{b_{n,n-j}\, c_{n-j,j-k}}{\lambda_{n-j}},\\
&\quad j_5(k) = \max\{0,k-1\} \text{ and } j_6(k) = \min\{m,k+1\},\\
\widetilde{e}_{h_{m+1},n,n-k} &= \sum_{j=j_6}^{j_7} \widehat{b}_{n,n-j}\, e_{h_{m+1},n-j,n-k},\\
&\quad j_6(k) = \max\{-m, k-m-1\} \text{ and } j_7(k) = \min\{m, k+m+1\}.
\end{aligned}
$$

For the Laguerre case we have that the coefficients $c_{n-j,j-k}$ *are defined as* $c_{n,1}=0, c_{n,0}=n, c_{n,-1}=-n(n+\alpha)$ *and for the Hermite case* $c_{n,1}=0, c_{n,0}=0, c_{n,-1}=n$

Proof. Let $\rho(z)\widehat{Q}_n(z)=\sum_{k=-n}^{m}\widehat{b}_{n,n+k}\,P_{n+k}(z)$ be the Fourier expansion of $\rho\widehat{Q}_n$ with respect to the orthogonal system of polynomials $\{P_n\}$, then

$$\widehat{b}_{n,n+k}=\int P_{n+k}(x)\,\widehat{Q}_n(x)\,\rho(x)d\mu(x)=\langle P_{n+k},\widehat{Q}_n\rangle_w;$$

from (3.18), if $-n\leq k\leq -m-1$ $\langle P_{n+k},\widehat{Q}_n\rangle_w=0$. For $-m\leq k\leq m$, the formula of $\widehat{b}_{n,n+k}$ in the lemma, is computed by substituting (3.42) and (3.15) in $\langle L_{n+k},\widehat{Q}_n\rangle_w$ and using the Laguerre or Hermite polynomials orthogonality.

To prove (3.48) we use relations (3.44), (3.47)

$$\begin{aligned}
h_{m+1}(z)\rho(z)\widehat{Q}_n(z) &= \sum_{j=-m}^{m}\widehat{b}_{n,n-j}\;h_{m+1}(z)\,P_{n-j}(z)\\
&= \sum_{j=-m}^{m}\widehat{b}_{n,n-j}\sum_{k=-m-1}^{m+1}e_{(h_{m+1},n-j,n-j-k)}\,P_{n-j-k}(z)\\
&= \sum_{j=-m}^{m}\widehat{b}_{n,n-j}\sum_{k=j-m-1}^{j+m+1}e_{h_{m+1},n-j,n-k}\,P_{n-k}(z)\\
&= \sum_{k=-2m-1}^{2m+1}\left(\sum_{j=\max\{-m,k-m-1\}}^{\min\{m,k+m+1\}}\widehat{b}_{n,n-j}\,e_{h_{m+1},n-j,n-k}\right)P_{n-k}(z)\\
&= \sum_{k=-2m-1}^{2m+1}\widetilde{e}_{h_{m+1},n,n-k}\,P_{n-k}(z).
\end{aligned}$$

We prove now (3.49). Note that from (3.15) we have that

$$z\,\widehat{Q}'_n(z)=\lambda_n\sum_{k=0}^{m}\frac{b_{n,n-k}}{\lambda_{n-k}}\;z\;(L_{n-k}(z))', \tag{3.51}$$

and from the structure relation for monic Laguerre polynomials [163, (5.1.8),(5.1.14)]

$$z\;(L_{n-k}(z))'=c_{n-k,0}\,L_{n-k}(z)+c_{n-k,-1}\,L_{n-k-1}(z).$$

By substituting this formula in (3.51) we get

$$z\widehat{Q}'_n(z)=\sum_{k=-1}^{m+1}\widetilde{c}_{n,n-k}\;L_{n-k}(z),$$

and from (3.43) we have (3.49).

In a similar way, for the Hermite case we substitute the structure relation for monic Hermite polynomials [163, (5.5.10),(5.5.6)] into relation (2.18) and from (3.43) we have (3.50). □

Proof. (of Theorem 3.4)
If $\widehat{Q}_n$ denotes the monic orthogonal polynomial with respect to $(\mathcal{L}, \mu)$, we have that

$$H(z)\widehat{Q}_n(z) = \sum_{k=-2m-1}^{n-1} \vartheta_{n,n-k}\widehat{Q}_{n-k}(z) + \vartheta_{n,0}.$$

Hence

$$\mathcal{L}\left[H\widehat{Q}_n\right](z) = \sum_{k=-2m-1}^{n-1} \vartheta_{n,n-k}\,\lambda_{n-k}\,P_{n-k}(z). \tag{3.52}$$

Suppose now that $\mathcal{L}$ is the Laguerre operator. Note that the left hand of (3.52) equals to

$$\begin{aligned}\mathcal{L}[H\widehat{Q}_n](z) &= H(z)\mathcal{L}[\widehat{Q}_n(z)] + \widehat{Q}_n(z)\mathcal{L}[H(z)] + 2H'(z)z\widehat{Q}'_n(z)\\ &= \lambda_n\,H(z)\,P_n(z) + h_{m+1}(z)\,\rho(z)\,\widehat{Q}_n(z) + 2\,\rho^2(z)z\widehat{Q}'_n(z),\end{aligned} \tag{3.53}$$

where $h_{m+1}(z) = z\rho'(z) + \mathcal{L}[R](z)$ is a polynomial of degree $m+1$.
From (3.49)

$$z\rho^2(z)\widehat{Q}'_n(z) = \sum_{j=-m-1}^{m+1} d_{n,n-j}\;\rho(z)\,P_{n-j}(z).$$

From (3.45) we obtain

$$z\rho^2(z)\widehat{Q}'_n(z) \quad = \quad \sum_{k=-2m-1}^{2m+1} \widetilde{e}_{h_{m+1},n,n-k}\,P_{n-k}(z). \tag{3.54}$$

From (3.48), (3.54) we have that (3.53) can be expressed as

$$\sum_{k=-2m-1}^{2m+1} \left(\lambda_n e_{H,n,n-k} + \widetilde{e}_{\rho,n,n-k} + \widetilde{e}_{h_{m+1},n,n-k}\right) P_{n-k}(z).$$

Identifying coefficients in (3.52) we have

$$H(z)\widehat{Q}_n(z) = \sum_{k=-2m-1}^{2m+1} \vartheta_{n-k,k}\widehat{Q}_{n-k}(z) + \vartheta_{n,0},$$

and from relation (3.18) we obtain (3.41)
Analogously, for the Hermite operator $\mathcal{L}$ we have that the right hand of (3.52) equals to

$$\begin{aligned}\mathcal{L}[H\widehat{Q}_n](z) &= H(z)\mathcal{L}[\widehat{Q}_n(z)] + \widehat{Q}_n(z)\mathcal{L}[H(z)] + 2H'(z)\widehat{Q}'_n(z)\\ &= \lambda_n\,H(z)\,P_n(z) + h_{m+1}(z)\,\rho(z)\,\widehat{Q}_n(z) + 2\,\rho^2(z)\widehat{Q}'_n(z),\end{aligned} \tag{3.55}$$

where $h_{m+1} = \rho' + \mathcal{L}[R]$ is a polynomial of degree $m+1$.
From (3.50)

$$2\rho^2(z)\widehat{Q}'_n(z) = \sum_{j=-m-1}^{m+1} 2d_{n,n-j}\;\rho(z)\,P_{n-j}(z).$$

From (3.45) we obtain

$$2\rho^2(z)\widehat{Q}_n'(z) \quad = \quad \sum_{k=-2m-1}^{2m+1} \widetilde{e}_{h_{m+1},n,n-k}\, P_{n-k}(z). \tag{3.56}$$

From (3.48), (3.56) we have that (3.55) can be expressed as

$$\sum_{k=-2m-1}^{2m+1} \left(\lambda_n e_{H,n,n-k} + \widetilde{e}_{\rho,n,n-k} + \widetilde{e}_{h_{m+1},n,n-k}\right) P_{n-k}(z).$$

Identifying coefficients in (3.52) we have

$$H(z)\widehat{Q}_n(z) = \sum_{k=-2m-1}^{2m+1} \vartheta_{n-k,k}\widehat{Q}_{n-k}(z) + \vartheta_{n,0},$$

and from relation (3.18) we obtain (3.41). □

3.5 Zero location and asymptotic behavior of the normalized polynomials

Asymptotic formulas for the zero of the largest modulus of the Laguerre and Hermite polynomials have been studied in detail in [163]. Regardless of the use of c_n in the preceding sections, in what follows we denote by c_n the zero of the largest modulus of the nth Laguerre polynomial or Hermite, from [163, (6.32.8)] we know that

$$c_n = \begin{cases} 4n & +O(n^{-1/3}), \quad \text{Laguerre polynomials,} \\ \sqrt{2n} & +O(n^{-1/6}), \quad \text{Hermite polynomials.} \end{cases} \tag{3.57}$$

Assume that $m \in \mathbb{N}$ and that $\mu \in \mathcal{P}_m(\Delta)$. From Theorem 3.1 we had that m is the least natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$.

Let $\{\zeta_n\}_{n=m+1}^{\infty}$ be a sequence of complex numbers, $m \in \mathbb{N}$ and assume that $\mu \in \mathcal{P}_m(\Delta)$. We will be interested in the monic normalized polynomials defined by $\widehat{\mathfrak{Q}}_n(z) = c_n^{-n}\ \widehat{Q}_n(c_n z)$. We complement the definition of the sequence $\{\mathfrak{Q}_n\}$ considering that henceforth $\mathfrak{Q}_n$ for each $n > m$ is the monic polynomial such that

$$\mathfrak{Q}_n(z) = c_n^{-n}\ Q_n(c_n z). \tag{3.58}$$

We say that $\{\mathfrak{Q}_n\}$, for $n > m$, is the sequence of normalized monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $\mathfrak{Q}_n(\zeta_n) = 0$.

In this section we study the zero location and asymptotic properties of the normalized monic orthogonal polynomials with respect to a Laguerre or Hermite differential operator.

3.5.1 Zero location

We begin this subsection by finding asymptotic bounds for the coefficients $b_{n,n-k}$ that define the polynomial $\widehat{Q}_n$.

LEMMA 3.5. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m[\Delta]$. Then for n large enough, there are constants C_ρ^L and C_ρ^H such that*

$$|b_{n,n-k}| = \frac{1}{\|L_{n-k}\|_w^2}\left|\int P_n(x)L_{n-k}(x)dw(x)\right| < \begin{cases} C_\rho^L\, n^k & \text{Laguerre case,} \\ C_\rho^H\, \sqrt{n^k} & \text{Hermite case,} \end{cases}$$

for $k = 1, \ldots, m$.

Proof. Let $\rho(x) = \sum_{j=1}^{m} \rho_j x^j$ and $\rho_+ = \max_{0 \le j \le m} |\rho_j|$. From the Cauchy–Schwarz inequality we have

$$\begin{aligned} |b_{n,n-k}| &\le \frac{\|P_n\|_\mu}{\|L_{n-k}\|_w^2}\sqrt{\int \rho(x)L_{n-k}^2(x)dw(x)} \le \frac{\|\rho L_{n-m}\|_\mu}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\int \rho(x)L_{n-k}^2(x)dw(x)} \\ &\le \frac{1}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\sum_{j=0}^{m}|\rho_j|\left|\int x^j\, L_{n-m}^2(x)dw(x)\right|}\sqrt{\sum_{j=0}^{m}|\rho_j|\left|\int x^j\, L_{n-k}^2(x)dw(x)\right|} \\ &\le \frac{\rho_+}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\sum_{j=0}^{m}\left|\int x^j\, L_{n-m}^2(x)dw(x)\right|}\sqrt{\sum_{j=0}^{m}\left|\int x^j\, L_{n-k}^2(x)dw(x)\right|}. \end{aligned} \tag{3.59}$$

We analyze separately the Laguerre and Hermite cases. Without loss of generality we can assume that $n > 2m$.

• *Laguerre case* ($L_n = L_n^\alpha$, *and* $dw(x) = x^\alpha e^{-x}dx$). From [152, (III.4.9) and (I.2.9)] we have the connection formula

$$L_{n-k}^\alpha(z) = \sum_{\nu=k}^{k+j}\binom{j}{\nu}\frac{(n-k)!}{(n-\nu)!}L_{n-\nu}^{\alpha+j}(z),$$

then

$$\begin{aligned} \int x^j\,\left(L_{n-k}^\alpha(x)\right)^2 dw(x) &= \int \left(L_{n-k}^\alpha(x)\right)^2 x^{\alpha+j}e^{-x}dx, \\ &= \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}\frac{(n-k)!}{(n-\nu)!}\int \left(L_{n-\nu}^{\alpha+j}(x)\right)^2 x^{\alpha+j}e^{-x}dx, \\ &= \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}(n-k)!\Gamma(n-\nu+j+\alpha+1), \\ &\le 2^j(n-k)!\Gamma(n-k+j+\alpha+1), \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{m}\int x^j\, L_{n-k}^2(x)dw(x) &\le (n-k)!\sum_{j=0}^{m}2^j\Gamma(n-k+j+\alpha+1), \\ &\le (2^{m+1}-1)(n-k)!\Gamma(n-k+m+\alpha+1). \end{aligned}$$

Hence, from (3.59), (3.16) and n large enough

$$\begin{aligned} |b_{n,n-k}| &\leq \frac{\rho_+(2^{m+1}-1)}{|\rho_m|}\sqrt{\frac{(n-m)!\Gamma(n+\alpha+1)\Gamma(n+m-k+\alpha+1)}{(n-k)!\Gamma^2(n-k+\alpha+1)}}, \\ &\leq \frac{\rho_+(2^{m+1}-1)}{|\rho_m|}\sqrt{\frac{(n+\alpha)^{k+m}}{(n-m)^{m-k}}} \leq \frac{\rho_+ 2^m(2^{m+1}-1)}{|\rho_m|}\, n^k. \end{aligned} \tag{3.60}$$

• *Hermite case* ($L_n = H_n,\ \Delta = \mathbb{R}$ *and* $dw(x) = e^{-x^2}dx$). By the symmetry property of the Hermite polynomials, if ν is an odd number

$$\int x^\nu\, H_{n-k}^2(x)dw(x) = 0.$$

Hence, from (3.59)

$$|b_{n,n-k}| \leq \frac{\rho_+}{|\rho_m|\,\|H_{n-k}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j\,H_{n-m}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j\,H_{n-k}\|_w^2},$$

where for all $x \in \mathbb{R}$, the symbol $\lfloor x\rfloor$ denote the largest integer less than or equal to x. As is well known (cf. [163, (5.5.6) and (5.5.8)]), the Hermite polynomials satisfy the recurrence relation

$$zH_n(z) = H_{n+1}(z) + \frac{n}{2}H_{n-1}(z),$$

from which we get by induction on j

$$z^j H_n(z) = \sum_{\nu=0}^{j}\sigma_{j,\nu}(n)H_{n+j-2\nu}(z), \tag{3.61}$$

where $\sigma_{j,\nu}(n)$ is a polynomial in n of degree equal to ν and leading coefficient $2^{-\nu}\binom{j}{v}$ (i.e. $\sigma_{j,\nu}(n) = 2^{-\nu}\binom{j}{v}n^\nu + \cdots$). Hence, from (3.16), for n large enough

$$\begin{aligned} \|x^j H_{n-k}\|_w^2 &= \sum_{\nu=0}^{j}\sigma_{j,\nu}^2(n-k)\|H_{n-k+j-2\nu}\|_w^2 \\ &= \frac{\sqrt{\pi}}{2^{n-k+j}}\left(\sum_{\nu=0}^{j}2^{2\nu}\sigma_{j,\nu}^2(n-k)(n-k+j-2\nu)!\right) \\ &\leq \frac{\sqrt{\pi}\,(n-k-j)!}{2^{n-k+j}}\left(\sum_{\nu=0}^{j}2^{2\nu}\sigma_{j,\nu}^2(n-k)(n-k+j)^{2j-2\nu}\right) \\ &\leq \frac{2\sqrt{\pi}\,(n-k-j)!(n-k)^{2j}}{2^{n-k+j}}\sum_{\nu=0}^{j}\binom{j}{v}^2 \leq \frac{2\sqrt{\pi}\,(n-k-j)!(n-k)^{2j}}{2^{n-k}}\binom{2j}{j}, \end{aligned}$$

with $j = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$, therefore

$$
\begin{aligned}
|b_{n,n-k}| &\leq \frac{\rho_+ \, 2^{n-k}}{\sqrt{\pi}|\rho_m| \, (n-k)!} \sqrt{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \|x^j \, H_{n-m}\|_w^2} \sqrt{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \|x^j \, H_{n-k}\|_w^2} \\
&\leq \frac{2m!\rho_+}{|\rho_m|} \sqrt{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (n-m)^{2j} \frac{(n-m-j)!}{(n-k)!}} \sqrt{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (n-k)^{2j} \frac{(n-k-j)!}{(n-k)!}} \\
&\leq \frac{2m!\rho_+}{|\rho_m|} \sqrt{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n-m)^{2j}}{(n-m-j)^{m+j-k}}} \sqrt{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(n-k)^{2j}}{(n-m-j)^{j}}} \\
&\leq \frac{2m!\rho_+}{|\rho_m|} \sqrt{8m(n-k)^{-\lfloor \frac{m}{2} \rfloor}} \sqrt{2m(n-k)^{\lfloor \frac{m}{2} \rfloor}} \, n^k = \frac{8m^2(m-1)!\rho_+}{|\rho_m|} \, n^k. \qquad (3.62)
\end{aligned}
$$

□

Notice that expressions (3.60) and (3.62) provide explicit formulas for the constants C_ρ^L and C_ρ^H in Lemma 3.5.

Denoting by Δ_c the interval $[0, 1]$ in the Laguerre case and $[-1, 1]$ in the Hermite case, we introduce the distance function $d_c(z) = \min_{x \in \Delta_c} |z - x|$.

LEMMA 3.6. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m[\Delta]$ and ζ such that $\widehat{\mathfrak{Q}}_n(\zeta) = 0$. Then for n large*

$$d_c(\zeta) < \varpi_c,$$

where

$$\varpi_c = \begin{cases} 1 + 2^{-1} \, C_\rho^L & \textit{Laguerre case}, \\ 1 + \sqrt{2} \, C_\rho^H & \textit{Hermite case}, \end{cases}$$

and the constants C_ρ^L and C_ρ^H are the same as in Lemma 3.5.

Proof. For each fixed $n > m$, we have that

$$\widehat{\mathfrak{Q}}_n(z) = \sum_{k=0}^{m} \frac{\lambda_n \, b_{n,n-k}}{c_n^k \lambda_{n-k}} \, \mathfrak{L}_{n,n-k}(z),$$

where λ_ν as in (3.8) and $\mathfrak{L}_{n,\nu}(z) = c_n^{-\nu} \, L_\nu(c_n z)$. It is straightforward, that the polynomials $\{\mathfrak{L}_{n,\nu}\}_{\nu=0}^{\infty}$ are the monic orthogonal polynomials with respect to the weight $w_n(x) = w(c_n \, x)$. Furthermore, the smallest compact interval that contains the zeros of $\mathfrak{L}_{n,\nu}, 0 \leq \nu \leq n$ is Δ_c.

From [155, Corollary 1], we have that if ζ is a zero of $\widehat{\mathfrak{Q}}_n$ then

$$d_c(\zeta) \leq 1 + \max_{1 \leq k \leq m} \left| \frac{\lambda_n \, b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| < 1 + 2 \max_{1 \leq k \leq m} \left| \frac{b_{n,n-k}}{c_n^k} \right|.$$

The proof of the lemma is completed using Lemma 3.5 and (3.57). □

In the sequel, we denote by $\Theta_c = \{z \in \mathbb{C} : d_c(z) \leq \varpi_c\}$.

3.5.2 n-root asymptotic behavior for $\widehat{\mathfrak{Q}}_n$ and $\widehat{\mathfrak{Q}}'_n$

We recall some preliminary results and notations that shall be used in this subsection. Let $\mathfrak{L}_n^\alpha(z) = \mathfrak{L}_{n,n}^\alpha(z) = c_n^{-n}\, L_n^\alpha(c_n z)$ and $\mathfrak{H}_n(z) = \mathfrak{H}_{n,n}(z) = c_n^{-n}\, H_n(c_n z)$ be the normalized monic Laguerre and Hermite polynomials.

From [147, Th. 4 and Th. 4'] we find that the limit distribution ν of the zero counting measure of the normalized Laguerre and Hermite polynomials is

$$d\nu_w(t) = \begin{cases} \dfrac{2}{\pi}\sqrt{\dfrac{1-t}{t}}, & t \in [0,1] \quad \text{Laguerre case,} \\ \dfrac{2}{\pi}\sqrt{1-t^2}, & t \in [-1,1] \quad \text{Hermite case.} \end{cases} \tag{3.63}$$

$$\lim_{n\to\infty} \left|\mathfrak{L}_n^{(\alpha)}(z)\right|^{\frac{1}{n}} = \frac{1}{e}\, |\psi(z)|\, e^{2\,\mathrm{Re}[1/\varphi(z)]} \tag{3.64}$$

$$\lim_{n\to\infty} |\mathfrak{H}_n(z)|^{\frac{1}{n}} = \frac{1}{2\sqrt{e}}\, |\varphi(z)|\, e^{\mathrm{Re}\,[z/\varphi(z)]}. \tag{3.65}$$

uniformly on compact subsets $K \subset \mathbb{C}\setminus\Delta_c$, where $\varphi(z) = z + \sqrt{z^2-1}$, $\psi(z) = 2z - 1 + 2\sqrt{z(z-1)}$. Here we choose the branch of the root of the functions for which $\varphi(\infty) = \infty$ and $\psi(\infty) = \infty$.

In a similar way, if d_n denotes the modulus of the largest zero of P_n, then $\mathfrak{P}_n(z) = d_n^{-n} P_n(d_n z)$ denotes the normalized monic orthogonal polynomials with respect to a measure $\mu \in \mathcal{P}_m(\Delta)$. From [147] it follows that

$$\lim_{n\to\infty} \frac{c_n}{d_n} = 1, \tag{3.66}$$

as well as

$$d\nu_\mu(t) = \begin{cases} \dfrac{2}{\pi}\sqrt{\dfrac{1-t}{t}}dt, & t \in [0,1] \quad \text{for} \quad \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ \dfrac{2}{\pi}\sqrt{1-t^2}dt, & t \in [-1,1] \quad \text{for} \quad \mu \in \mathcal{P}_m(\mathbb{R}). \end{cases}$$

Hence, from [153] we have

$$\lim_{n\to\infty} |\mathfrak{P}_n(z)|^{\frac{1}{n}} = \frac{1}{e}\, |\psi(z)|\, e^{2\,\mathrm{Re}[1/\varphi(z)]}, \quad \text{for} \quad \mu \in \mathcal{P}_m(\mathbb{R}_+), \tag{3.67}$$

$$\lim_{n\to\infty} |\mathfrak{P}_n(z)|^{\frac{1}{n}} = \frac{1}{2\sqrt{e}}\, |\varphi(z)|\, e^{\mathrm{Re}\,[z/\varphi(z)]}, \quad \text{for} \quad \mu \in \mathcal{P}_m(\mathbb{R}). \tag{3.68}$$

uniformly on compact subsets $K \subset \mathbb{C}\setminus\Delta_c$.

Let γ be any Jordan closed curve. According to the Jordan's curve theorem, γ divides the complex plane in two regions, we shall denote by $int(\gamma)$ the region of the complex plane which does not contain the ∞ point.

In this section we prove the following theorem

THEOREM 3.5. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m[\Delta]$. Then*

$$\lim_{n\to\infty} \left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\widehat{\mathfrak{Q}}'_n(z)\right|^{\frac{1}{n}} = \begin{cases} \frac{1}{e}\, |\psi(z)|\, e^{2\,\mathrm{Re}[1/\varphi(z)]} & \textit{Laguerre case,} \\ \frac{1}{2\sqrt{e}}\, |\varphi(z)|\, e^{\mathrm{Re}\,[z/\varphi(z)]} & \textit{Hermite case.} \end{cases}$$

uniformly on compact subsets K on $\mathbb{C}\setminus\Theta_c$.

We prove some preliminary lemmas

LEMMA 3.7. *Suppose that $\Xi \subset \mathbb{C}$ is a compact subset and $K \subset \mathbb{C} \setminus \Xi$, is compact. If the accumulation points of the zeros $\{z_{k,n}\}_{k=0}^{n}$ of a family of monic polynomials $\{R_n\}_{n\in\mathbb{N}}$ are contained on Ξ, then the limits points of the sequence $\left\{\dfrac{R_n'}{R_n}\right\}_{n\in\mathbb{N}}$ in the uniform convergence norm are non zero on K.*

Proof. Let ψ be a limit point on K in the uniform convergence norm of the sequence $\left\{\dfrac{1}{n}\sum_{k=1}^{n}\dfrac{1}{z-z_{k,n}}\right\}_{n\in\mathbb{N}} = \left\{\int \dfrac{d\nu_n(\varepsilon)}{z-\varepsilon}\right\}_{n\in\mathbb{N}}$ where $\nu_n = \dfrac{1}{n}\sum_{k=1}^{n}\delta_{k,n}$ are probabilities measures and $\delta_{k,n}$ are Dirac measures with mass 1 at the zero $z_{k,n}$ and assume that $\psi \equiv 0$ on K, we have then that for some subsequence $\{n_k\}_{k\in\mathbb{N}}$, $\psi(z) = \lim_{k\to\infty}\int \dfrac{d\nu_{n_k}(\varepsilon)}{z-\varepsilon}$. By Helly's selection theorem [153, page 3], we can select from $\{\nu_{n_k}\}_{k\in\mathbb{N}}$ a weak star convergent subsequence, with support contained on Ξ. Denote by ν this limit point.

Since $\{\nu_{n_k}\}_{k\in\mathbb{N}}$ are probability measures it follows that ν is not the null measure and hence, $\int \dfrac{d\nu(\varepsilon)}{z-\varepsilon} = \psi(z)$ is not null, which is a contradiction. $\square$

LEMMA 3.8. *Suppose that $\Xi \subset \mathbb{C}$ is a compact subset and $K \subset \mathbb{C}\setminus\Xi$ is compact. If the set of the accumulation points of the zeros $\{z_{k,n}\}_{k=0}^{n}$ of a family of monic polynomials $\{R_n\}_{n\in\mathbb{N}}$ is contained on Ξ, then there exists a constant k_0 (depending on K) such that*

$$k_0 \leq \frac{1}{n}\left|\frac{R_n'(z)}{R_n(z)}\right|, \quad \forall z \in K.$$

Proof. Assume that the inequality is not true. Then it is possible to find a subsequence $\{R_{n_k}(z_{n_k})\}_{k\in\mathbb{N}}$, $\{z_{n_k}\}_{k\in\mathbb{N}} \subset K$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\frac{R_{n_k}'(z_{n_k})}{R_{n_k}(z_{n_k})}\right| = 0.$$

Consider a compact $K' \subset \mathbb{C}\setminus\Xi$ such that $K \subset K'$. As the family $\left\{\dfrac{1}{n_k}\dfrac{R_{n_k}'}{R_{n_k}}\right\}_{k\in\mathbb{N}}$ is bounded on K', by the Montel's theorem, the family is compact on the interior of K', denoted by $int[K']$. Hence, there exist a subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ such that $\psi(z) = \dfrac{1}{n_m}\sum_{k=1}^{n_m}\dfrac{1}{z-z_{k,n_m}}$ uniformly on $K \subset int[K']$ and $\psi(z_0) = 0$, for $z_0 \in K$, by Lemma 3.7 ψ is non null on K. By Hurwitz's theorem we deduce that we can find a m_0 such that $\dfrac{1}{n_{k_m}}\sum_{k=1}^{n_{k_m}}\dfrac{1}{z-z_{k,n_m}}$ has a zero $\forall m > m_0$ in some neighborhood of z_0, but this is not possible, since the set of the accumulation points of the zeros of $\dfrac{1}{n_m}\sum_{k=1}^{n_m}\dfrac{1}{z-z_{k,n_m}}$ is contained on Ξ. $\square$

LEMMA 3.9. *Let $K \subset \mathbb{C}\setminus\Theta_c$ be a compact subset, then*

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)}\right|}\leq 1, \quad \forall z\in K, \tag{3.69}$$

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}'_n(z)}\right|}\leq 1, \quad \forall z\in K. \tag{3.70}$$

Proof. Consider fist the Laguerre case and inequality (3.69). From (3.8) we have

$$\left|\frac{\lambda_n P_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\leq\left|\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\left|c_nz\frac{\widehat{Q}''_n(c_nz)}{\widehat{Q}'_n(c_nz)}\right|+|1+\alpha-c_nz|\left|\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right| \tag{3.71}$$

From Lemma 3.6 we have that the zeros of $\widehat{Q}'_n(c_nz)$ are located on the interior of Θ_c, we have then

$$\left|c_nz\frac{\widehat{Q}''_n(c_nz)}{\widehat{Q}'_n(c_nz)}\right|\leq\sum_{k=1}^{n-1}\left(1+\left|\frac{z'_{k,n}}{z-z'_{k,n}}\right|\right)\leq\frac{d+n}{d}, \tag{3.72}$$

$$|1+\alpha-c_nz|\left|\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\leq\sum_{k=1}^{n}\left(1+\left|\frac{1+\alpha}{c_nz-c_nz_{k,n}}\right|+\left|\frac{z_{k,n}}{z-z_{k,n}}\right|\right)\leq\frac{4nd+|1+\alpha|+4n}{4d}, \tag{3.73}$$

$$\left|\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\leq\frac{1}{4d}, \tag{3.74}$$

where $\{z_{k,n}\}_{k=0}^n$,$\{z'_{k,n}\}_{k=0}^n$ are the zeros of $\widehat{Q}_n(c_nz),\widehat{Q}'_n(c_nz)$ respectively and $d=dist(K,\Theta_c)$.
From (3.71),(3.72),(3.73) and (3.74) we deduce that

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)}\right|}\leq 1, \forall z\in K,$$

and this prove (3.69) for the Laguerre case. Inequality (3.70) follows immediately from (3.69) and Lemma 3.8.

We proceed to prove (3.69) for the Hermite case. From (3.8) and the Gauss Lucas Theorem we have

$$\left|\frac{\lambda_n P_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\leq\left|\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\left|\frac{\widehat{Q}''_n(c_nz)}{\widehat{Q}'_n(c_nz)}\right|+\left|2c_nz\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|, \tag{3.75}$$

we have that

$$\left|\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\leq\sum_{k=1}^{n}\frac{1}{|z-z_{k,n}|}\leq\frac{n}{d}, \tag{3.76}$$

$$\left|2c_nz\frac{\widehat{Q}'_n(c_nz)}{\widehat{Q}_n(c_nz)}\right|\leq 2\left(n+\sum_{k=1}^{n}\left|\frac{z_{k,n}}{z-z_{k,n}}\right|\right)\leq 2\left(n+\frac{n}{d}\right), \tag{3.77}$$

$$\left|2c_nz\frac{\widehat{Q}''_n(c_nz)}{\widehat{Q}'_n(c_nz)}\right|\leq\sum_{k=1}^{n-1}2\left(1+\left|\frac{z'_{k,n}}{z-z'_{k,n}}\right|\right)\leq\frac{2n(d+1)}{d}, \tag{3.78}$$

where $d = dist(K, \Theta_c)$. From (3.75), (3.76),(3.77) and (3.78) we deduce (3.69) for the Hermite case. Analogously, Inequality (3.70) is an immediate consequence of (3.69). $\square$

To prove the reverse inequality, we shall need some previous lemmas.

LEMMA 3.10. *Let γ_1, γ_2 be two Jordan curves such that γ_1 encloses Θ_c, $int[\gamma_1] \subset int[\gamma_2], \gamma_1 \bigcap \gamma_2 = \emptyset$ and γ_2, for the Laguerre case, satisfies that $\frac{1}{2} + t\imath \in int(\gamma_2)$, where t is a real parameter large enough and for the Hermite case $t\imath \in int(\gamma_2)$. Let us define the set $\mathcal{R} = int[\gamma_2] \setminus int[\gamma_1]$. If there exists an infinite subsequence $\{n_k\}_{k=0}^{\infty}$ such that*

$$\lim_{k\to\infty} \frac{\widehat{Q}''_{n_k}(c_{n_k} z)}{\widehat{Q}'_{n_k}(c_{n_k} z)} = f_L(z), \tag{3.79}$$

$$\lim_{k\to\infty} \frac{\widehat{Q}''_{n_k}(c_{n_k} z)}{\widehat{Q}'_{n_k}(c_{n_k} z)} = f_H(z), \tag{3.80}$$

uniformly on $\mathcal{R}$, then $f_L(z) \neq 1, \forall z \in \mathcal{R}$ and $f_H(z) \neq z, \forall z \in \mathcal{R}$ according to the case.

Proof. First, let us consider the Laguerre case. Suppose that

$$\lim_{k\to\infty} \frac{\widehat{Q}''_{n_k}(c_{n_k} z)}{\widehat{Q}'_{n_k}(c_{n_k} z)} = 1, \quad \forall z \in \mathcal{R}.$$

Since $\left\{ \dfrac{\widehat{Q}''_{n_k}(c_{n_k} z)}{\widehat{Q}'_{n_k}(c_{n_k} z)} \right\}_{k=0}^{\infty}$ is a sequence of analytic functions we have

$$\lim_{k\to\infty} \int \frac{\widehat{Q}''_{n_k}(c_{n_k} z)}{\widehat{Q}'_{n_k}(c_{n_k} z)} dz = \lim_{k\to\infty} \frac{1}{c_{n_k}} \int \frac{\widehat{\mathcal{Q}}''_{n_k}(z)}{\widehat{\mathcal{Q}}'_{n_k}(z)} dz = z, \forall z \in \mathcal{R},$$

and this gives

$$\lim_{k\to\infty} \frac{1}{4} \log \left(\widehat{\mathcal{Q}}'_{n_k}(z) \right)^{\frac{1}{n_k}} = z, \forall z \in \mathcal{R},$$

which implies

$$\lim_{k\to\infty} \left| \widehat{\mathcal{Q}}'_{n_k}(z) \right|^{\frac{1}{n_k}} = e^{4\operatorname{Re}(z)}, \forall z \in \mathcal{R}. \tag{3.81}$$

From (3.66),(3.67) and (3.69) we deduce that

$$\lim_{k\to\infty} \left| \widehat{\mathcal{Q}}'_{n_k}(z) \right|^{\frac{1}{n_k}} \geq e^{2\operatorname{Re} z - 2\operatorname{Re}\sqrt{z(z-1)} + \log|2z - 1 + 2\sqrt{z(z-1)}| - 1}, \tag{3.82}$$

uniformly on $\mathcal{R}$. Relations (3.81), (3.82) imply

$$4\operatorname{Re} z \geq 2\operatorname{Re} z - 2\operatorname{Re}\sqrt{z(z-1)} + \log|2z - 1 + 2\sqrt{z(z-1)}| - 1, \quad \forall z \in \mathcal{R},$$

but it is easy to see that this last inequality is not valid at the point $z = \frac{1}{2} + t\imath \in \mathcal{R}$, for $t > 0$ large enough, which is a contradiction.

Repeating the same reasoning as above we can arrive at the following inequality for the Hermite case

$$2\operatorname{Re} z^2 \geq \log|z + \sqrt{z^2 - 1}| + \operatorname{Re} z(z - \sqrt{z^2 - 1}) - \frac{1}{2} - \log 2,$$

but it is easy to see that this last inequality is not valid at the point $z = t\imath \in \mathcal{R}$, for $t > 0$ large enough, and this is a contradiction. $\square$

LEMMA 3.11. *Denote by $\{z'_{k,n}\}_{k=0}^{n}$ the zeros of the polynomial $\widehat{Q}'_n$. Then, for every compact $K \subset \mathbb{C} \setminus \Theta_c$ there exist a positive integer number n_0 and $\varepsilon > 0$ such that for all $n > n_0$*

$$\left| z \sum_{k=1}^{n} \frac{1}{c_n z - c_n z'_{k,n}} - \left(z - \frac{1-\alpha}{c_n} \right) \right| \geq \varepsilon, \quad \textit{Laguerre case}, \tag{3.83}$$

$$\left| \frac{1}{c_n} \sum_{k=1}^{n} \frac{1}{c_n z - c_n z'_{k,n}} - 2z \right| \geq \varepsilon, \quad \textit{Hermite case}, \tag{3.84}$$

uniformly on K.

Proof. Let us define

$$f_{n,L}(z) = z \sum_{k=1}^{n-1} \frac{1}{c_n z - c_n z'_{k,n}} - \left(z - \frac{1-\alpha}{c_n} \right), \quad n > m \quad \text{(Laguerre case)} \tag{3.85}$$

$$f_{n,H}(z) = \frac{1}{c_n} \sum_{k=1}^{n-1} \frac{1}{c_n z - c_n z'_{k,n}} - 2z, \quad n > m \quad \text{(Hermite case).} \tag{3.86}$$

Let $\mathcal{R}$ be defined as in Lemma 3.10 with γ_2 satisfying the additional condition $K \subset \mathcal{R}$. It is easy to see that $\{f_n\}_{n=m+1}^{\infty}$ is a sequence of analytic functions which is uniformly bounded on the closure of $\mathcal{R}$ (denoted by $\overline{\mathcal{R}}$) . Hence, from the Montel's theorem, the family $\{f_n\}_{n=0}^{\infty}$ is compact on the interior of $\overline{\mathcal{R}}$ and hence, it contains a convergent subsequence $\{f_{n_j}\}_{j=0}^{\infty}$ on K, let $\psi_L = \lim_{j\to\infty} f_{n_j,L}$, $\psi_H = \lim_{j\to\infty} f_{n_j,H}$ be the respective limits of the Laguerre and Hermite.

Assume that (3.83) (or (3.83)) is not valid. Then, there exists a sequence $\{z_{n_j}\}_{j\in\mathbb{N}} \subset K$ such that

$$\lim_{j\to\infty} f_{n_j,L}(z_{n_j}) = 0,$$
$$\lim_{j\to\infty} f_{n_j,H}(z_{n_j}) = 0.$$

As K is compact, there exists a convergent subsequence $\{z_{n_{j_m}}\}_{m=0}^{\infty} \subset \{z_{n_j}\}_{j=0}^{\infty}$ such that $\lim_{m\to\infty} z_{n_{j_m}} = z_0 \in K$.

As $\{f_{n_j,L}\}_{j=0}^{\infty}$ (or $\{f_{n_j,H}\}_{j=0}^{\infty}$) is a sequence of analytic functions then we have that $\psi_L(z_0) = 0$ ($\psi_H(z_0) = 0$) and from Lemma 3.10 we deduce that ψ_L is not the null function on $\mathcal{R}$ and therefore, on K for the Laguerre case. By the same reason, ψ_H is non null for the Hermite case. Hence, by the Hurwitz's theorem, for every neighborhood U_{z_0} of z_0 we can find a m_0 such that if $m > m_0$ then $f_{n_{j_m},L}(z_{n_{j_m}})$ (or $f_{n_{j_m},H}(z_{n_{j_m}})$) has a zero of the same multiplicity as ψ_L(or ψ_H) on z_0.

From the identities

$$\frac{1}{\lambda_n} \left(\frac{\widehat{Q}'_n(c_n z)}{\widehat{Q}_n(c_n z)} \right) c_n f_{n,L}(z) = \frac{P_n(c_n z)}{\widehat{Q}_n(c_n z)} \quad \text{Laguerre case,}$$

$$\frac{1}{\lambda_n} \left(\frac{\widehat{Q}'_n(c_n z)}{\widehat{Q}_n(c_n z)} \right) c_n f_{n,H}(z) = \frac{P_n(c_n z)}{\widehat{Q}_n(c_n z)} \quad \text{Hermite case,}$$

we deduce that for some n, $P_n(c_n z)$ has a zero on K which is impossible. □

LEMMA 3.12. *If $K \subset \mathbb{C} \setminus \Theta_c$ is compact subset then*

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)}\right|} \geq 1, \quad \forall z \in K, \tag{3.87}$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}'_n(z)}\right|} \geq 1, \quad \forall z \in K. \tag{3.88}$$

Proof. From Lemma 3.8 it is possible to find constants k_l, k_h such that

$$\begin{aligned} k_l &\leq \left|\frac{\widehat{Q}'_n(c_n z)}{\widehat{Q}_n(c_n z)}\right|, \quad \forall z \in K,, \quad \text{Laguerre case,} \\ k_h &\leq \frac{1}{c_n}\left|\frac{\widehat{Q}'_n(c_n z)}{\widehat{Q}_n(c_n z)}\right|, \quad \forall z \in K,, \quad \text{Hermite case,} \end{aligned} \tag{3.89}$$

Hence, from (3.8),(3.89) and Lemma 3.11 we have that there exist constants k_l^*, k_h^* such that uniformly on K

$$\begin{aligned} \frac{k_l^*\, c_n}{\lambda_n} &\leq \frac{c_n}{\lambda_n}\left|\frac{\widehat{Q}'_n(c_n z)}{\widehat{Q}_n(c_n z)}\right|\left|z\frac{\widehat{Q}''_n(c_n z)}{\widehat{Q}'_n(c_n z)} - \left(z - \frac{1-\alpha}{c_n}\right)\right| = \left|\frac{P_n(c_n z)}{\widehat{Q}_n(c_n z)}\right|, \quad \text{Laguerre case,} \\ \frac{k_h^*\, c_n}{\sqrt{\lambda_n}} &\leq \frac{c_n}{\lambda_n}\left|\frac{\widehat{Q}'_n(c_n z)}{\widehat{Q}_n(c_n z)}\right|\left|\frac{1}{c_n}\frac{\widehat{Q}''_n(c_n z)}{\widehat{Q}'_n(c_n z)} - 2z\right| = \left|\frac{P_n(c_n z)}{\widehat{Q}_n(c_n z)}\right|, \quad \text{Hermite case,} \end{aligned}$$

and this proves (3.87). Inequality (3.88) follows from (3.87) and Lemma 3.8. □

Proof. (*of Theorem 3.5*)
Theorem 3.5 is an immediate consequence of (3.66),(3.67),(3.68) and Lemmas 3.9 and 3.12. □

3.5.3 The polynomial $\mathfrak{Q}_n$

In this section we the asymptotic behavior of the zeros and a n root asymptotic formula for the polynomial $\mathfrak{Q}_n$ on some compact subsets of the complex plane. We denote for $z \in \mathbb{C}$, $\mathfrak{D}(z) = \sup_{x\in\Theta_c} |z-x|$ and $\mathfrak{d}(z) = \inf_{x\in\Theta_c} |z-x|$.

3.5.4 Asymptotic behavior of the zeros

Some basic properties of the zeros of $\mathfrak{Q}_n$ follow directly from (3.1), (3.2). For example, the multiplicity of the zeros of $\mathfrak{Q}_n$ is at most 3, a zero of multiplicity 3 is also a zero of $\mathfrak{P}_n$ and a zero of multiplicity 2 is a critical point of $\widehat{\mathfrak{Q}}_n$. In the next two theorems, we prove conditions for the boundedness of the zeros of $\mathfrak{Q}_n$ and determine its asymptotic behavior.

THEOREM 3.6. *Let $\mu \in \mathcal{P}_m[\Delta]$, where $m \in \mathbb{N}$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta \in \mathbb{C}$, then:*

1. *For every $d > 1$ there is a positive number N_d, such that $\{z \in \mathbb{C} : \mathfrak{Q}_n(z) = 0\} \subset \{z \in \mathbb{C} : |z| \leq \mathfrak{D}(\zeta) + d\}$ whenever $n > N_d$.*

2. *If $\mathfrak{d}(\zeta) > 2$, the zeros of $\mathfrak{Q}_n$ can not accumulate on Θ_c and for n sufficiently large are simple.*

Proof. As $\mathfrak{Q}_n(z) = 0$ then $\widehat{\mathfrak{Q}}_n(z) = \widehat{\mathfrak{Q}}_n(\zeta_n)$. From Gauss–Lucas theorem (cf. [157, §2.1.3]), it is known that the critical points of $\widehat{\mathfrak{Q}}_n$ are on the convex hull of his zeros and from Lemma 3.6 the zeros of the polynomials $\{\widehat{\mathfrak{Q}}_n\}$ are located on Θ_c. Hence, from the *bisector theorem* (see [157, §5.5.7]) $|z| \leq \mathfrak{D}(\zeta_n) + 1$ and the first part of the Theorem is established.

To verify the second statement of the theorem, note that if z is a zero of Q_n, from (3.58) we get

$$\prod_{k=1}^{n} \left| \frac{z - \widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}} \right| = 1. \tag{3.90}$$

On the one hand, let $\mathcal{V}_\varepsilon(\Theta_c)$ be the ε–neighborhood of Θ_c defined as $\mathcal{V}_\varepsilon(\Theta_c) = \{z \in \mathbb{C} : \mathfrak{d}(z) < \epsilon\}$. On the other hand, as $\lim\limits_{n\to\infty} \zeta_n = \zeta$, then for all $\varepsilon > 0$ there is a $N_\varepsilon > 0$ such that $|\mathfrak{d}(\zeta_n) - \mathfrak{d}(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$.

If $\mathfrak{d}(\zeta) > 2$, let us choose $\varepsilon = \varepsilon_\zeta = \dfrac{1}{2}\left(\mathfrak{d}(\zeta) - 2\right)$ and suppose that there is a $z_0 \in \mathcal{V}_{\varepsilon_\zeta}(\Theta_c)$ such that $\mathfrak{Q}_n(z_0) = 0$ for some $n > N_{\varepsilon_\zeta}$. Hence

$$\prod_{k=1}^{n} \left| \frac{z_0 - \widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}} \right| < \left(\frac{2 + \varepsilon_\zeta}{\mathfrak{d}(\zeta_n)} \right)^n < 1,$$

which is a contradiction with (3.90), hence $\{z \in \mathbb{C} : \mathfrak{Q}_n(z) = 0\} \bigcap \mathcal{V}_{\varepsilon_n}(\Theta_c) = \varnothing$ for all $n > N_{\varepsilon_\zeta}$, i.e. the zeros of $\mathfrak{Q}_n$ can not accumulate on $\mathcal{V}_{\varepsilon_\zeta}(\Theta_c)$.

From (3.58) is straightforward that a multiple zero of $\mathfrak{Q}_n$ is also a critical point of $\widehat{\mathfrak{Q}}_n$. But, from Lemma 3.6 and the Gauss–Lucas theorem the critical point of $\widehat{\mathfrak{Q}}_n$ are contained on Θ_c, where we have that for n sufficiently large the zeros of $\mathfrak{Q}_n$ are simple. □

THEOREM 3.7. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m[\Delta]$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex number with limit $\zeta \in \mathbb{C} \setminus \Theta_c$, then the accumulation points of zeros of $\{\mathfrak{Q}_n\}_{n=m+1}^{\infty}$ are located on the set $E = \mathcal{E}(\zeta) \bigcup \Theta_c$, where $\mathcal{E}(\zeta)$ is the curve*

$$\mathcal{E}(\zeta) := \{z \in \mathbb{C} : \Psi(z) = \Psi(\zeta)\}, \tag{3.91}$$

where $\Psi(z) = |\psi(z)|\, e^{2\,\mathrm{Re}[1/\varphi(z)]}$ for the Laguerre case and $\Psi(z) = |\varphi(z)|\, e^{\mathrm{Re}\,[z/\varphi(z)]}$ for the Hermite case. If $\mathfrak{d}(\zeta) > 2$ then $E = \mathcal{E}(\zeta)$.

Proof. From 1 of Theorem 3.6 we have that the zeros of $\mathfrak{Q}_n$ are located in a compact set. From (3.58) the zeros of $\mathfrak{Q}_n$ satisfy the equation

$$\left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \left|\widehat{\mathfrak{Q}}_n(\zeta_n)\right|^{\frac{1}{n}}. \tag{3.92}$$

If $z \in \mathbb{C} \setminus \Theta_c$, taking limit when $n \to \infty$, from 2 of Theorem 3.6, and using 1 of Theorem 3.5 in both sides of (3.92), we have that the zeros of the sequence of polynomials $\{\mathfrak{Q}_n\}_{n=m+1}^{\infty}$ cannot accumulate outside the set

$$\{z \in \mathbb{C} : \Psi(z) = \Psi(\zeta)\} \bigcup \Theta_c.$$

The assertion for $\mathfrak{d}(\zeta) > 2$ is straightforward from 2. of Theorem 3.6. □

Asymptotic behavior

THEOREM 3.8. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\Delta)$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta \in \mathbb{C} \setminus \Theta_c$, then*

$$\frac{\mathfrak{Q}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} \underset{n}{\rightrightarrows} 1, \tag{3.93}$$

uniformly on compact subsets of $\{z \in \mathbb{C} : |z| > \mathfrak{D}(\zeta) + 1\}$.

Proof. Let K be a compact subset of $\{z \in \mathbb{C} : |z| > \mathfrak{D}(\zeta) + 1\}$. From (3.58)

$$\frac{\mathfrak{Q}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} = \frac{Q_n(z)}{\widehat{Q}_n(z)} = 1 - \frac{\widehat{Q}_n(c_n\zeta_n)}{\widehat{Q}_n(c_n z)}\,.$$

Hence, in order to prove (3.93) it is sufficient to show that

$$\frac{\widehat{Q}_n(c_n\zeta_n)}{\widehat{Q}_n(c_n z)} \underset{n}{\rightrightarrows} 0, \tag{3.94}$$

uniformly on a compact subsets $K \subset \{z \in \mathbb{C} : |z| > \mathfrak{D}(\zeta) + 1\}$.

As $\lim\limits_{n\to\infty} \zeta_n = \zeta$, for all $\varepsilon > 0$ there is a N_ε such that $|\mathfrak{D}(\zeta_n) - \mathfrak{D}(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$. Let $\widehat{x}_{n,1}, \widehat{x}_{n,2}, \ldots, \widehat{x}_{n,n}$ be the n zeros of $\widehat{\mathfrak{Q}}_n$ and $d_{K,\zeta} = \inf\limits_{\substack{z\in K\\ |w|=\mathfrak{D}(\zeta)+1}} |z-w|$, where K is a compact subset of $\{z \in \mathbb{C} : |z| > \mathfrak{D}(\zeta) + 1\}$.

Now, let us choose $\varepsilon = \dfrac{1}{2}\, d_{K,\zeta}$

$$\left|\frac{\widehat{Q}_n(c_n\zeta_n)}{\widehat{Q}_n(c_n z)}\right| = \frac{\prod_{k=1}^{n} |\zeta_n - \widehat{x}_{n,k}|}{\prod_{k=1}^{n} |z - \widehat{x}_{n,k}|} < \left(\frac{\Delta(\zeta_n)}{\Delta(\zeta) + d_{K,\zeta}}\right)^n < \left(\frac{\Delta(\zeta) + \varepsilon}{\Delta(\zeta) + d_{K,\zeta}}\right)^n < 1, \quad z \in K.$$

This inequality is equivalent to the uniform convergence of (3.94) on a compact subset K of $\{z \in \mathbb{C} : |z| > \mathfrak{D}(\zeta) + 1\}$. The statement (3.93) is then a direct consequence of (3.94). □

From Theorem 3.8 and Theorem 3.5 we obtain

COROLLARY 3.1. *With the same conditions of the Theorem 3.8, the following limits hold uniformly on each compact subsets of* $\{z \in \mathbb{C} : |z| > \mathfrak{D}(\zeta) + 1\}$

$$\lim_{n\to\infty} \left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \begin{cases} \frac{1}{e}\, |\psi(z)|\, e^{2\,\mathrm{Re}[1/\varphi(z)]} & \textit{Laguerre case,} \\ \frac{1}{2\sqrt{e}}\, |\varphi(z)|\, e^{\mathrm{Re}\,[z/\varphi(z)]} & \textit{Hermite case.} \end{cases}$$

It would be of interest to obtain a formula for the strong asymptotic behavior of the sequence $\{\mathfrak{P}_n\}_{n=0}^{\infty}$ on compact sets of $\mathbb{C} \setminus \Delta$ in order to apply for Theorem 3.8 a technique similar to the considered in Theorem 2.2. Taking into account the results obtained in Theorems 3.8 and 2.2, we conjecture that

CONJECTURE 1. *Let $\{\zeta_n\}_{n=m+1}^{\infty}$ be a sequence of complex numbers with limit $\zeta \in \mathbb{C}\setminus\Delta$, $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$ and $\{Q_n\}_{n=m+1}^{\infty}$ the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $Q_n(\zeta_n) = 0$, then:*

1. *Uniformly on compact subsets of* $\Omega = \{z \in \mathbb{C} : |\Psi(z)| > |\Psi(\zeta)|\}$,

$$\frac{\mathfrak{Q}_n(z)}{\mathfrak{Q}_n(z)} \underset{n}{\rightrightarrows} 1.$$

2. *Uniformly on compact subsets of* $\Omega = \{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\}$

$$\frac{\mathfrak{Q}_n(z)}{\mathfrak{Q}_n(\zeta_n)} \underset{n}{\rightrightarrows} -1,$$

Chapter 4

Orthogonal polynomials with respect to a class of differential operators

4.1 Introduction

In this chapter we consider orthogonal polynomials with respect to a linear exactly solvable differential operator. We analyze the uniqueness and zero location of these polynomials. An interesting phenomena occurring in this kind of orthogonality is the existence of operators for which the associated sequence of orthogonal polynomials reduces to a finite set. For a given operator, we find a classification of the measures for which it is possible to guarantee the existence of an infinite sequence of orthogonal polynomials, in terms of a linear system of difference equations with varying coefficients. Also, for the case of a first order differential operator, we locate the zeros and establish the strong asymptotic behavior of these polynomials.

Some of the techniques used here could be extended in some degree to the general case of linear homogeneous differential operators with polynomial coefficients including the class of Heine–Stieltjes operators as well as the lowering and raising operators with polynomial coefficients, but we will not dwell into this.

The chapter is organized as follows. In Section 4.2 we present connections between this type of orthogonality and some inner products and classify the exactly solvable operators for which this concept of orthogonality reduces to an inner product. In Section 4.3 we give necessary and sufficient conditions for the normality of an index n. The analysis of the existence of infinite sequences of polynomials $\{Q_n\}_{n=m}^{\infty}$, for some positive m, with $deg[Q_n] = n$, in terms of a linear system of difference equations with varying coefficients is done in Section 4.4. In Section 4.5, we study the location of the zeros for the polynomials Q_n and in Section 4.6, for a first order differential operator, we obtain a curve which contains the accumulation points of the zeros of the polynomials giving also the strong asymptotic behavior of the polynomials. The results of this chapter have been submitted for consider for publication in [22].

4.2 The inner product classification

In Section 1.3 of the introduction of this thesis we have presented the relation of orthogonality with respect to a differential operator to some inner products.

Let us consider now the bilinear form on $\mathbb{P}$

$$[Q, P] = \int \mathcal{L}^{(M)}[Q(x)]P(x)d\mu(x). \tag{4.1}$$

In general, it is not possible to state that the bilinear form (4.1) defines an inner product. The following theorem characterizes the exactly solvable operators and the measures μ for which (4.1) defines an inner product.

The following theorem characterizes the exactly solvable operators and the measures μ for which (4.1) defines an inner product.

THEOREM 4.1. *Let $\mathcal{L}^{(M)}$ be an exactly solvable and μ a positive Borel measure. Then, a necessary and sufficient condition for* (4.1) *to be an inner product is that:*

- *$\mathcal{L}^{(M)}$ is a Bochner-Krall operator and μ is the measure such that the polynomial eigenfunctions of the operator form a system of orthogonal polynomials.*
- *$\mathcal{L}^{(M)}$ has positive eigenvalues.*

In such a case, the sequence of monic polynomials $\{Q_n\}_{n=0}^{\infty}$ orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$ coincides with the monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ with respect to the measure μ.

Proof. Suppose that the relation (4.1) defines an inner product. We have then that (4.1) is symmetric, taking into account (1.2) and (1.3) we have,

$$\int \mathcal{L}^{(M)}[Q(x)]P(x)d\mu(x) = \langle \mathcal{L}^{(M)}[Q(x)]\sigma, P(x)\rangle = [Q, P]$$
$$= [P, Q] = \int \mathcal{L}^{(M)}[P(x)]Q(x)d\mu(x) = \langle \mathcal{L}^{(M)}[P]\sigma, Q\rangle;$$

that is,

$$\langle \mathcal{L}^{(M)}[Q]\sigma, P\rangle = \langle \mathcal{L}^{(M)}[P]\sigma, Q\rangle,$$

and from [98, Theorem 2.4(ii) implies i)] we have that $\mathcal{L}^{(M)}$ is a Bochner-Krall operator and μ is the measure with respect to which the polynomial eigenfunctions of the operator form a system of orthogonal polynomials.

The second condition follows from the fact that $[.,.]$ must define a positive definite bilinear form.

The converse implication is straightforward.

The assertion that the sequence of monic polynomials $\{Q_n\}_{n=0}^{\infty}$ coincides with the monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ with respect to the measure μ follows from the fact that $\mathcal{L}^{(M)}$ is an exactly solvable operator with positive eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ which implies that condition (1.9) is equivalent to solving

$$\mathcal{L}^{(M)}[Q_n] = \lambda_n P_n,$$

from where we deduce that $Q_n = P_n$. □

4.3 Necessary and sufficient conditions for the normality of an index

In this section we give necessary and sufficient conditions for the normality of an index n for the class of exactly solvable operators. As $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ is an exactly solvable operator, following (1.9), it is not difficult to see that the monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(M)}, \mu)$ associated to an index n are linear combinations of a monic polynomial solution of

$$\mathcal{L}^{(M)}[y] \quad = \quad \lambda_n P_n, \tag{4.2}$$

and a monic polynomial solution of

$$\mathcal{L}^{(M)}[y] \quad = \quad 0. \tag{4.3}$$

Here P_n denotes the n-th monic orthogonal polynomials with respect to μ and λ_n is the coefficient associated to the factor x^n of the polynomial $\mathcal{L}^{(M)}[x^n]$, which is $\lambda_n = \sum_{k=0}^{M} \rho_{k,k} \frac{n!}{(n-k)!}$, where by convention $\frac{n!}{(n-k)!} = 0$ when $k > n$. In the sequel we shall assume that λ_n will denote this coefficient.

Before we state the results of the section we show with an example that in general we do not have normality of an index for the class of operators that we consider.

EXAMPLE 4.3.1. *[Second order differential operator] Suppose that $M = 2$ and define $\mathcal{L}[f] = f'' - 2xf' + 2f, f \in \mathbb{P}$. Notice that the eigenfunctions of this operator are the Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ with eigenvalues $\lambda_n = 2(1-n)$ and that $\mathcal{L}[x] = 0$. Consider the measure $d\mu(x) = \frac{e^{-x^2}dx}{x^2+1}$ supported on $\mathbb{R}$ and denote by $\{P_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to μ. Notice that if $n > 3$ the polynomial P_n can be expanded in the basis $\{H_k\}_{k=0}^{n}$ as*

$$P_n(x) = H_n(x) + \alpha_{n-1}H_{n-1}(x) + \alpha_{n-2}H_{n-2}(x), \quad \alpha_{n-k} = \frac{\int P_n(x)H_{n-k}(x)e^{-x^2}dx}{\sqrt{\int H_{n-k}^2(x)e^{-x^2}dx}}; k = 1, 2, \tag{4.4}$$

from where we deduce that the monic orthogonal polynomial Q_n with respect to $(\mathcal{L}, \mu)$ for the index n can be described as

$$Q_n(x) = H_n(x) + \frac{\lambda_n \alpha_{n-1}}{\lambda_{n-1}} H_{n-1}(x) + \frac{\lambda_n \alpha_{n-2}}{\lambda_{n-2}} H_{n-2}(x) + cx, \quad c \in \mathbb{R}, \quad n > 3.$$

By a similar argument, expanding P_0, P_1, P_2, P_3 in terms of H_0, H_1, H_2, H_3 we have that the solutions to (4.2), (4.3) *give that if $n \leq 3$ then $Q_0(x) = 1, Q_1(x) = Q_2(x) = Q_3(x) = x$; therefore, we have normality only for $n \leq 3$.*

Remark 1. *We correct here [8, Ex. 1]. There, it is stated that any exactly solvable operator for which $\rho_M \equiv 1$ and $\rho_k \not\equiv 0, 0 \leq k < M$ satisfies the conditions of [8, Th. 3].*

For the operator of example 4.3.1 of the present paper, $\mathcal{L}[f] = f'' - 2xf' + 2f$. The function $\rho_{0,1}$ defined in [8, Th. 3] (according with the notation employed in that paper) simplifies to $\rho_0 + \rho_1 = 2 - 2x$ and this function has a zero on supp(μ). This particular operator does not satisfy the conditions given in [8, Th. 3]; therefore, that theorem cannot be applied for exactly solvable operators in general.

In order to provide a necessary and sufficient condition for the normality of an index, we introduce some auxiliary notation and prove some preliminaries lemmas. In the sequel, let Δ_n be the determinant of the Hankel matrix defined by the moments $\mu_0, \ldots, \mu_{2n}$ of the measure μ. Define $\Delta_{0,0} = \mu_0$ and denote by $\Delta_{n,i}, 0 \leq i \leq n$, the determinant of the following matrix with column $i+1$ deleted

$$\begin{pmatrix} \mu_0 & \cdots & \mu_n \\ & \vdots & \\ \mu_{n-1} & \cdots & \mu_{2n-1} \end{pmatrix}.$$

Consider the infinite upper triangular matrix $A = (a_{i,j})$ with entries

$$a_{i,j} = \sum_{k=j-i}^{\min(M,j-1)} \rho_{k,i+k-j} \frac{(j-1)!}{(j-1-k)!}, \quad i \le j, \tag{4.5}$$

and set $A_n = (a_{i,j})_{\substack{1\le i\le n \\ 1\le j\le n}}$.

Let $q_n(x) = \sum_{j=0}^{n} \alpha_{n,j} x^j$ be a generic polynomial of degree n. By $\mathfrak{a}_{n+1} = (\alpha_{n,0}, \dots, \alpha_{n,n})^t$ we denote the column vector of the coefficients of q_n.

LEMMA 4.1. *Let μ be a positive Borel measure on the real line and $\{P_n\}_{n=0}^{\infty}$ the associated sequence of monic orthogonal polynomials. Let $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ be an exactly solvable operator, where $\rho_k(x) = \sum_{j=0}^{k} \rho_{k,j} x^j$. Then* (4.2) *can be expressed as*

$$A_{n+1}\, \mathfrak{a}_{n+1} \quad = \quad \lambda_n \mathfrak{b}_{n+1}, \tag{4.6}$$

and (4.3) *can be expressed as*

$$A_{n+1}\, \mathfrak{a}_{n+1} = 0, \tag{4.7}$$

where $\mathfrak{b}_{n+1} = (\beta_{n,0}, \dots, \beta_{n,n})^t$ with

$$\beta_{n,i} = \Delta_{n,i} \Delta_{n,n}^{-1}, \quad 0 \le i \le n,$$

is the column vector of the coefficients of P_n.

Proof. Let $q_n(x) = \sum_{j=0}^{n} \alpha_{n,j} x^j$. We have that

$$\begin{aligned}
\mathcal{L}^M \left[\sum_{j=0}^{n} \alpha_j x^j \right] &= \sum_{k=0}^{M} \left(\sum_{u=0}^{k} \rho_{k,u} x^u \sum_{j=0}^{n} \alpha_{n,j} x^{j-k} \frac{j!}{(j-k)!} \right) \qquad (4.8) \\
&= \sum_{k=0}^{M} \sum_{j=k}^{\min(k,n)} \sum_{u=0}^{k} \rho_{k,u} \frac{j!}{(j-k)!} x^{u+j-k} \alpha_{n,j} \\
&= \sum_{k=0}^{M} \sum_{i=0}^{n} \sum_{j=\max(k,i)}^{\min(n,i+k)} \left(\rho_{k,i+k-j} \frac{j!}{(j-k)!} \alpha_{n,j} \right) x^i \\
&= \sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{k=j-i}^{\min(M,j)} \left(\rho_{k,i+k-j} \frac{j!}{(j-k)!} \alpha_{n,j} \right) x^i .
\end{aligned}$$

From Heine's formula for the monic orthogonal polynomials, we have

$$P_n(x) = \Delta_{n,n}^{-1} \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ & \vdots & \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix}. \tag{4.9}$$

Therefore, (4.2) or (4.3) can be expressed in matrix form as

$$A_{n+1}\,\mathfrak{a}_{\mathfrak{n}+\mathfrak{1}} \quad = \quad \lambda_n \mathfrak{b}_{\mathfrak{n}+\mathfrak{1}},$$

and

$$A_{n+1}\,\mathfrak{a}_{\mathfrak{n}+\mathfrak{1}} = 0,$$

respectively. □

LEMMA 4.2. *Let* $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ *be an exactly solvable operator,* $\rho_k(x) = \sum_{i=0}^{k} \rho_{k,i} x^i$ *and* A_{n+1} *the matrix whose entries are defined by* (4.5). *Then* $a_{j+1,j+1}$ *is the coefficient of* x^j *of the polynomial* $\mathcal{L}^{(M)}[x^j]$.

Proof. We have

$$\begin{aligned} \mathcal{L}^{(M)}[x^j] &= \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k} x^j \\ &= \sum_{k=0}^{M} \rho_k(x) \frac{j!}{(j-k)!} x^{j-k} \\ &= \sum_{k=0}^{\min(M,j)} \sum_{i=0}^{k} \rho_{k,i} \frac{j!}{(j-k)!} x^{i+j-k}. \end{aligned}$$

From this expression we obtain that if $i = k$ then the coefficient of x^j in $\mathcal{L}^{(M)}[x^j]$ is

$$\sum_{k=0}^{\min(M,j)} \rho_{k,k} \frac{j!}{(j-k)!},$$

which corresponds to the coefficient $a_{j+1,j+1}$ in (4.5) of the matrix A_{n+1}. □

From the preceding lemmas we deduce a necessary and sufficient condition for the normality of an index n.

THEOREM 4.2. *Let* μ *be a positive Borel measure on the real line and* $\{P_n\}_{n=0}^{\infty}$ *the associated sequence of monic orthogonal polynomials. Let* $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ *be an exactly solvable differential operator. Then an index* $n \in \mathbb{Z}_+$ *is normal if and only if either*

i) $deg[\mathcal{L}^{(M)}[x^k]] = k, \quad \forall k: \quad 0 \leq k \leq n,$

or

ii) There exist indexes $n_1, \dots, n_k; 0 \le n_1 \le \dots \le n_k \le n$, such that $deg[\mathcal{L}^{(M)}[x^{n_j}]] < n_j, 1 \le j \le k$,

1) if $k \ge 1$, then $\{\mathcal{L}^{(M)}[x^{n_1}], \dots, \mathcal{L}^{(M)}[x^{n_k}]\}$ has $n_k - n_1$ linearly independent vectors,

2) if $n_k < n$, then the moments of the measure μ satisfy the relation

$$\sum_{j=0}^{n-n_k-1} \gamma_{n-n_k-j} \Delta_{n,n-j} \quad \neq \quad \Delta_{n,n_k}, \tag{4.10}$$

where $\{\gamma_i\}_{i=1}^{n-n_k}$ are such that

$$\begin{pmatrix} 1 & \gamma_1 & & \cdots & \gamma_{n-n_k} \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & & & & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ a_{n_k+2,n_k+2} & a_{n_k+2,n_k+3} & & & \vdots \\ \vdots & \vdots & & & \\ & & \cdots & a_{n,n} & a_{n,n+1} \\ & 0 & \cdots & 0 & a_{n+1,n+1} \end{pmatrix} \tag{4.11}$$

and B is the matrix

$$B = \begin{pmatrix} a_{n_k+1,n_k+2} & & \cdots & & \\ a_{n_k+2,n_k+2} & a_{n_k+2,n_k+3} & & & \vdots \\ & \vdots & & & \\ & & \cdots & a_{n,n} & a_{n,n+1} \\ 0 & & \cdots & 0 & a_{n+1,n+1} \end{pmatrix}.$$

Proof. We assume that $n \ge 1$, otherwise we have that $n = 0$ is normal and there is nothing to prove. Suppose that the index n is normal. This is equivalent to saying that the following alternatives for (4.2) and (4.3) hold.

a) Equation (4.2) has an unique monic polynomial solution.

b) Equation (4.3) has an unique non zero monic polynomial solution and $\lambda_n \neq 0$.

c) Equation (4.3) has an unique non zero monic polynomial solution and $\lambda_n = 0$.

If we have alternative $a)$, then Lemma 4.1 gives that this statement is equivalent to $Ker[A_{n+1}] = \{0\}$, hence the elements of the diagonal of the matrix A_{n+1} are non null.

By Lemma 4.2 we obtain that $deg[\mathcal{L}^{(M)}[x^k]] = k, \forall k : 0 \le k \le n$, that is, we have $i)$.

Suppose we have alternatives $b)$ or $c)$. From the statement that equation (4.3) has an unique non zero monic polynomial solution we deduce that there exist indexes $n_1, \dots, n_k; 0 \le n_1 \le \dots \le n_k \le n$, such that $deg[\mathcal{L}^{(M)}[x^{n_j}]] < n_j, 1 \le j \le k$. Denote by $0 \le n_1 \le \dots \le n_k \le n$ the indexes for which $deg[\mathcal{L}^{(M)}[x^{n_j}]] < n_j$. If $k > 1$ we partition the matrix A_{n+1} in blocks as

$$A_{n+1} = \begin{pmatrix} \widetilde{B}_3 \\ \widetilde{B}_2 \\ \widetilde{B}_1 \end{pmatrix},$$

where $\widetilde{B}_3$ is the block of A_{n+1} formed by its first n_1 rows, $\widetilde{B}_2$ contains the rows $n_1 + 1, \dots, n_k$ of A_{n+1} and $\widetilde{B}_1$ has the rows $n_k + 1, \dots, n + 1$ of A_{n+1}. When $n_1 = 0$ we have that

$$A_{n+1} = \begin{pmatrix} \widetilde{B}_2 \\ \widetilde{B}_1 \end{pmatrix}.$$

When $n_k = n$ we have that

$$A_{n+1} = \begin{pmatrix} \widetilde{B}_3 \\ \widetilde{B}_2 \end{pmatrix}.$$

Suppose we have alternative $b)$. Then equation (4.3) has an unique non zero monic polynomial solution; hence, there exist indexes $n_1, \ldots, n_k; 0 \leq n_1 \leq \ldots \leq n_k \leq n$, such that $deg[\mathcal{L}^{(M)}[x^{n_j}]] < n_j, 1 \leq j \leq k$. If $k > 1$ and $deg[\mathcal{L}^{(M)}[x^{n_j}]] < n_j, \forall j = 1, \ldots, k$, by Lemma 4.2, $a_{n_j+1,n_j+1} = 0$; hence, $rank[\widetilde{B}_1] = n - n_k$ and $rank[\widetilde{B}_3] = n_1$ (if $\widetilde{B}_3$ is empty, by convention $rank[\widetilde{B}_3] = 0$). As (4.3) has an unique non null monic polynomial solution and Lemma 4.1 we deduce that $dim[Ker[A_{n+1}]] = 1$; therefore, $rank[A_{n+1}] = n$. Hence if we denote by v_i the i–th row of A_{n+1}, then we have

$$\begin{aligned} n &= rank\left[\{v_i\}_{i=1}^{n_1} \bigcup \{v_i\}_{i=n_1+1}^{n_k} \bigcup \{v_i\}_{i=n_k+1}^{n+1}\right] \\ &\leq rank\left[\{v_i\}_{i=1}^{n_1}\right] + rank[\{v_i\}_{i=n_1+1}^{n_k}] + rank[\{v_i\}_{i=n_k+1}^{n+1}] = n_1 + rank[\{v_i\}_{i=n_1+1}^{n_k}] + n - n_k, \end{aligned}$$

which implies that $rank[\widetilde{B}_2] = n_k - n_1$; that is, the number of independent rows of the block $\widetilde{B}_2$ is $n_k - n_1$ and in virtue of Lemmas 4.1, 4.2, this is equivalent to saying that $\{\mathcal{L}^{(M)}[x^{n_1}], \ldots, \mathcal{L}^{(M)}[x^{n_k}]\}$ has $n_k - n_1$ linearly independent vectors and we have $1)$ of $ii)$.

Notice that for the case $k = 1$ we have an unique index n_1 such that $\mathcal{L}^{(M)}[x^{n_1}] = 0$, which evidently gives that $\{\mathcal{L}^{(M)}[x^{n_1}]\}$ has 0 linearly independent vectors.

Consider now the statement of $b)$ that $\lambda_n \neq 0$ or equivalently, $deg[\mathcal{L}^{(M)}[x^n]] = n$. From Lemma 4.2 we have necessarily that $n_k < n$. Since equation (4.3) has an unique non zero monic polynomial solution and the index n is normal by hypothesis, then (4.2) has no solution, from Lemma 4.1 we deduce that the system

$$A_{n+1}\, \mathfrak{a}_{n+1} = \lambda_n \mathfrak{b}_{n+1},$$

is necessarily incompatible. Let us multiply the above relation on both sides by the matrix Γ, where

$$\Gamma = \begin{pmatrix} I & & & 0 & \\ & 1 & \gamma_1 & \cdots & \gamma_{n-n_k} \\ & 0 & 1 & 0 & \cdots \\ 0 & & & & \\ & & & \vdots & \\ & 0 & & & 1 \end{pmatrix},$$

and the $\{\gamma_i\}_{i=1}^{n-n_k}$ are as in (4.11). Then

$$\Gamma\, A_{n+1}\, \mathfrak{a}_{n+1} = \lambda_n\, \Gamma\, \mathfrak{b}_{n+1}. \tag{4.12}$$

Note that in the obtained system, row $n_k + 1$ is zero; hence, system (4.12) is incompatible if and only if the component $n_k + 1$ in vector $\mathfrak{b}_{n+1}$ satisfies that

$$\sum_{j=0}^{n-n_k-1} \gamma_{n-n_k-j} \Delta_{n,n-j} \neq \Delta_{n,n_k},$$

and we obtain 2) of ii). Conversely, if 2) and 1) of ii) hold then we have again the statement b) which is equivalent to the normality of n.

Finally, suppose that alternative c) is the case. Lemma 4.1 gives that systems (4.6) and (4.7) are the same. If $k = 1$ then we have that $dim[Ker[A_{n+1}]] = 1$ and we are done. Assume that $k > 1$ then $n_k = n$ and

$$A_{n+1} = \begin{pmatrix} \widetilde{B}_3 \\ \widetilde{B}_2 \end{pmatrix}.$$

As the solution to (4.3) is non zero and unique we have that $dim[Ker[A_{n+1}]] = 1$, therefore, $rank[A_{n+1}] = n$ which implies that $rank[\widetilde{B}_2] = n_k - n_1$; that is, the number of linearly independent rows of the block $\widetilde{B}_2$ is $n_k - n_1$ and in virtue of Lemmas 4.1, 4.2, this is equivalent to saying that $\{\mathcal{L}^{(M)}[x^{n_1}], \ldots, \mathcal{L}^{(M)}[x^{n_k}]\}$ has $n_k - n_1$ linearly independent vectors and we have 1) of ii). It is not difficult to see that when $k = 1$ we have that $\{\mathcal{L}^{(M)}[x^{n_1}]\}$ has 0 linearly independent vectors. The converse implication is straightforward. $\square$

It is not difficult to see that the condition i) obtained in Theorem 4.2 is equivalent to affirming that

$$\{\mathcal{L}^{(M)}[1], \ldots, \mathcal{L}^{(M)}[x^n]\}, \tag{4.13}$$

is linearly independent which is also equivalent to saying that this set is a Markov system. In [8, Th 1] it was proved that if (4.13) forms a Markov system then we have normality of an index for linear homogeneous differential operators in general.

It seems natural to conjecture that for a general homogeneous linear differential operator a necessary and sufficient condition could be that, either (4.13) is a Markov system or if $k \geq 1$ then

$$\{\mathcal{L}^{(M)}[x^{n_1}], \ldots, \mathcal{L}^{(M)}[x^{n_k}]\},$$

has $n_k - n_1$ linearly independent functions on the support of the measure μ, plus some additional conditions on the moments of the measure.

4.4 Existence and uniqueness of polynomial solutions of degree n

An interesting phenomena that occurs in this type of orthogonality is the existence of operators and measures for which the associated sequence of orthogonal polynomials reduces to a finite set. A very simple example can be constructed to illustrate this.

EXAMPLE 4.4.1. *[First order differential operator] Let $\mathcal{L}[f](x) = xf', f \in \mathbb{P}$, and consider any positive Borel measure μ supported on a compact subset of $\mathbb{R}_+$. According to* (1.9)*, the orthogonal polynomial Q_n with respect to $\mathcal{L}$ associated to the index n is defined by*

$$\int xQ_n'(x)x^k d\mu(x) = 0, \quad \forall k \leq n-1.$$

But this is only possible if and only if $Q_n' \equiv 0$. Hence the sequence $\{Q_n\}_{n=0}^{\infty}$ reduces to a constant.

The preceding example shows that the sequence of polynomials $\{Q_n\}_{n=0}^{\infty}$ orthogonal with respect to the operator $\mathcal{L}[f](x) = xf', f \in \mathbb{P}$, and any positive Borel measure supported on a compact subset of $\mathbb{R}_+$ reduces to a constant. In this section we analyze necessary and sufficient conditions on the pair $(\mathcal{L}^{(M)}, \mu)$ for the existence and uniqueness of infinite sequences of orthogonal polynomials. We shall need the following preliminary lemma.

LEMMA 4.3. *Let $\{\widehat{P}_n\}_{n=0}^{\infty}$, $deg[\widehat{P}_n] = n$, be a sequence of monic polynomials and $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ an exactly solvable differential operator on $\mathbb{P}$. Then the following conditions are equivalent:*

i) $deg[\mathcal{L}^{(M)}[x^n]] = n, \quad \forall n \geq 0.$

ii) For every $n \in \mathbb{Z}_+$ *there exists an unique monic polynomial* $\widehat{Q}_n$ *such that*

$$\mathcal{L}^{(M)}[\widehat{Q}_n] = \lambda_n \widehat{P}_n.$$

iii) $Ker[\mathcal{L}^{(M)}] = \{0\}.$

iv) $\lambda_n = \sum_{j=0}^{M} \rho_{j,j} \frac{n!}{(n-j)!} \neq 0, \quad \forall n \geq 0.$

Proof. $i) \Leftrightarrow ii)$

Suppose that $deg[\mathcal{L}^{(M)}[x^n]] = n, \forall k \geq 0$. Then we have that for every fixed $n_0 \geq 0$, $\left\{\mathcal{L}^{(M)}[x^n]\right\}_{n=0}^{n_0}$ is a basis of $\mathbb{P}_{n_0}$. Hence, it is possible to find $\{\alpha_n\}_{n=0}^{n_0}$ such that $\widehat{P}_{n_0}(x) = \sum_{k=0}^{n_0} \alpha_{n_0,k}\mathcal{L}^{(M)}[x^k]$ and thus, by construction we have that there exists an unique monic polynomial $\widehat{Q}_{n_0}(x) = \sum_{k=0}^{n_0} \alpha_k x^k$ such that $\mathcal{L}^{(M)}[\widehat{Q}_n] = \lambda_n \widehat{P}_n$ holds and we get $ii)$.

Suppose now that for some index n_0 we have that $deg[\mathcal{L}^{(M)}[x^{n_0}]] < n_0$. From this fact and the hypothesis that $\mathcal{L}^{(M)}$ is exactly solvable, every polynomial Q_{n_0} of degree less or equal to n_0 satisfies that $\mathcal{L}^{(M)}[\widehat{Q}_{n_0}]$ is a polynomial of degree less than n_0 and hence it cannot satisfy $\mathcal{L}^{(M)}[\widehat{Q}_n] = \lambda_n \widehat{P}_n$. That is $ii) \Rightarrow i)$; therefore, $i) \Leftrightarrow ii)$.

$ii) \Leftrightarrow iii)$.

Assume that $ii)$ holds. As $\widehat{Q}_n$ is unique, for every non negative integer we have that $Ker[\mathcal{L}^{(M)}] = \{0\}$; that is, $ii) \Rightarrow iii)$. The converse implication is straightforward.

$i) \Leftrightarrow iv)$.

This follows from the fact that the coefficient associated to the factor x^n in $\mathcal{L}^{(M)}[x^n]$ is equal to

$$\sum_{j=0}^{M} \rho_{j,j} \frac{k!}{(k-j)!}.$$

□

We characterize now the exactly solvable operators for which we can guarantee the existence and uniqueness of an infinite sequence of orthogonal polynomials $\{Q_n\}_{n=0}^{\infty}$, such that each polynomial Q_n has degree equal to n.

THEOREM 4.3. *Let* μ *be a positive Borel measure on the real line and* $\{P_n\}_{n=0}^{\infty}$ *the associated sequence of monic orthogonal polynomials and* $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ *an exactly solvable operator with* $\rho_k(x) = \sum_{j=0}^{k} \rho_{k,j} x^j$. *Then there exists an unique sequence of monic polynomials* $\{Q_n\}_{n=0}^{\infty}$*, each polynomial* Q_n *of degree equal to* n*, and orthogonal with respect to* $(\mathcal{L}^{(M)}, \mu)$*, if and only if any of the statements of Lemma 4.3 hold.*

Proof. It follows from Lemma 4.3 by taking $\left\{\widehat{P}_n\right\}_{n=0}^{\infty} = \{P_n\}_{n=0}^{\infty}$ in $ii)$. □

A natural question then arises. What happens if any of the conditions of Lemma 4.3 does not hold?. It is not difficult to see that from the expression of λ_n as a polynomial in n given in $iv)$ of Lemma 4.3, only for a finite number of values this relation will not be valid. Let us denote by S the set of such indexes. In this case it is also possible to give necessary and sufficient conditions on the measure μ in order to have an infinite sequence $\{Q_n\}_{n\notin S}$ for which each monic polynomial Q_n has degree n. The following theorem characterizes such measures in terms of a finite set of difference equations with given initial conditions.

THEOREM 4.4. *Let μ be a positive Borel measure on the real line and $\{P_n\}_{n=0}^{\infty}$ the associated sequence of monic orthogonal polynomials. Let $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ be an exactly solvable differential operator and $\rho_k(x) = \sum_{j=0}^{k} \rho_{k,j} x^j$. Suppose that condition iv) of Lemma 4.3 is not satisfied and denote by $S = \{n_1, \dots, n_k\}$ the set of indexes for which that condition does not hold. Then, there exists a sequence of monic polynomials $\{Q_n\}_{n\notin S}$ orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$ if and only if the moments of the measure μ satisfy the system*

$$\begin{aligned} \sum_{v=-M}^{n_1} \left(\sum_{k=\max(-v,0)}^{M} \sum_{i=\max(0,v)}^{\min(n_1,v+k)} (-1)^{i+n_1} \frac{n!}{(n-k)!} \Delta_{n_1,i} \rho_{k,v-i+k} \right) \mu_{n+v} &= 0, \\ &\vdots \\ \sum_{v=-M}^{n_k} \left(\sum_{k=\max(-v,0)}^{M} \sum_{i=\max(0,v)}^{\min(n_k,v+k)} (-1)^{i+n_k} \frac{n!}{(n-k)!} \Delta_{n_k,i} \rho_{k,v-i+k} \right) \mu_{n+v} &= 0, \end{aligned} \tag{4.14}$$

where $n_k \in S$ and $n \notin S$. Moreover, if $\mu_0, \dots, \mu_{2n_k-1}$ are the moments of some positive measure supported on a subset of $\mathbb{R}$ satisfying (4.14)*, then for $n > n_k$ the system* (4.14) *defines a linear system of difference equations with varying coefficient and with initial conditions $\mu_0, \dots, \mu_{2n_k-1}$.*

Proof. By definition, the set $\{Q_k\}_{k=0}^{n}, n \notin S$, exists if and only if for every $n \notin S$ it is possible to find coefficients $\{\alpha_k\}_{k=0}^{n}$ such that

$$P_n(x) = \sum_{k\notin S} \alpha_k \mathcal{L}^{(M)}[x^k].$$

As $\mathcal{L}^{(M)}$ is exactly solvable the preceding condition is equivalent to

$$span[\{P_k\}_{k=0}^{n}] = span[\{\mathcal{L}^{(M)}[x^k]\}_{k=0}^{n}], \quad k, n \notin S, \tag{4.15}$$

and (4.15) is equivalent to saying that there exist coefficients $\{\beta_k\}_{k=0}^{n}$ such that

$$\mathcal{L}^{(M)}[x^n] = \sum_{k\notin S} \beta_k P_k(x), \quad n \notin S,$$

and this condition is satisfied if and only if μ satisfies the finite system of equations

$$\begin{aligned} \int \mathcal{L}^{(M)}[x^n] P_{n_1}(x) d\mu(x) &= 0, \\ &\vdots \\ \int \mathcal{L}^{(M)}[x^n] P_{n_k}(x) d\mu(x) &= 0, \end{aligned} \tag{4.16}$$

for all $n \notin S$ and $n_j \in S, j = 1, \dots, k$. Substituting in (4.16) Heine's formula (4.9) for the monic orthogonal polynomials we obtain

$$
\begin{array}{rcl}
\displaystyle\int \sum_{k=0}^{M} \frac{n!}{(n-k)!} \begin{vmatrix} \mu_0 & \cdots & \mu_{n_1} \\ & \vdots & \\ \mu_{n_1-1} & \cdots & \mu_{2n_1-1} \\ \displaystyle\sum_{j=0}^{k} \rho_{k,j} x^{n+j-k} & \cdots & \displaystyle\sum_{j=0}^{k} \rho_{k,j} x^{n+j-k+n_1} \end{vmatrix} d\mu(x) & = & 0, \\
& \vdots & \\
\displaystyle\int \sum_{k=0}^{M} \frac{n!}{(n-k)!} \begin{vmatrix} \mu_0 & \cdots & \mu_{n_k} \\ & \vdots & \\ \mu_{n_k-1} & \cdots & \mu_{2n_k-1} \\ \displaystyle\sum_{j=0}^{k} \rho_{k,j} x^{n+j-k} & \cdots & \displaystyle\sum_{j=0}^{k} \rho_{k,j} x^{n+j-k+n_k} \end{vmatrix} d\mu(x) & = & 0.
\end{array}
$$

Commuting the integral and the summation symbols, expanding the determinant by minors and doing some change of indexes we have

$$
\begin{array}{rcl}
\displaystyle\sum_{k=0}^{M} \sum_{i=0}^{n_1} \sum_{u=-k}^{0} (-1)^{i+n_1} \frac{n!}{(n-k)!} \Delta_{n_1,i} \rho_{k,u+k} \mu_{n+u+i} & = & 0, \\
& \vdots & \\
\displaystyle\sum_{k=0}^{M} \sum_{i=0}^{n_k} \sum_{u=-k}^{0} (-1)^{i+n_k} \frac{n!}{(n-k)!} \Delta_{n_k,i} \rho_{k,u+k} \mu_{n+u+i} & = & 0,
\end{array}
$$

which is equivalent to (4.14).

Consider now that $\mu_0, \dots, \mu_{2n_k-1}$ are the moments of some positive measure supported on a subset of $\mathbb{R}$ satisfying (4.14). It is not difficult to see that for $n > n_k$ system (4.14) defines a linear system of difference equations with varying coefficient and with initial conditions $\mu_0, \dots, \mu_{2n_k-1}$. $\square$

For a given operator $\mathcal{L}^{(M)}$, we denote the class of positive Borel measures with support contained in $\mathbb{R}$ which satisfy system (4.14) as $\Xi_{\mathcal{L}^{(M)}}$. Note that $\Xi_{\mathcal{L}^{(M)}}$ does not necessarily reduce to the empty set. We show some examples of the set $\Xi_{\mathcal{L}^{(M)}}$.

EXAMPLE 4.4.2. *Consider the first order linear differential operator* $\mathcal{L}[f](x) = xf'(x) - f(x), f \in \mathbb{P}$. *Note that* $\mathcal{L}[x^n] = (n-1)x^n$. *Hence the set of indexes n for which $iv)$ of Lemma 4.3 is not fulfilled reduces to* $n = 1$. *Then* (4.14) *reads*

$$
\begin{array}{rcl}
\mu_0 & = & c \in \mathbb{R}^+, \\
\mu_1 & = & c \in \mathbb{R}, \\
\mu_0 \mu_{n+1} - \mu_1 \mu_n & = & 0, \quad n > 1.
\end{array}
$$

EXAMPLE 4.4.3. *Consider the Euler–Cauchy operator* $\mathcal{L}^{(M)}[f](x) = \displaystyle\sum_{k=1}^{M} a_k x^k f^{(k)}(x)$, *where* $a_k \in \mathbb{R}$ *are* *such that the polynomial* $p(n) = \sum_{k=1}^{M} \frac{n!}{(n-k)!} a_k$ *does not have roots for* $n > 0$*. Then, we have that* $\mathcal{L}^M[x^n] = p(n)x^n$*. Let us assume that* $p(n)$ *has no integer roots for* $n > 0$*. Then, system* (4.14) *reduces to,*

$$\mu_n = 0, \quad n \geq 1,$$

which implies that $\mu \equiv 0$*. Hence, the set* $\Xi_{\mathcal{L}^{(M)}}$ *is empty.*

EXAMPLE 4.4.4. *Let* $\mathcal{L}_H[f](x) = f''(x) - 2\,x\,f'(x), f \in \mathbb{P}$ *be the Hermite operator. Then* (4.14) *is*

$$\begin{aligned} \mu_0 &= c, \quad c \in \mathbb{R}^+, \\ \mu_1 &= 0, \\ 2\mu_n - (n-1)\mu_{n-2} &= 0, \quad n \geq 2, \end{aligned}$$

which is the difference equation that defines the measure $c\,\mu_H$*, where* $d\mu_H(x) = e^{-x^2}dx$*.*

In a similar way, for the Laguerre and Jacobi operators $\mathcal{L}_L, \mathcal{L}_{(\alpha,\beta)}$, respectively, we obtain that $\Xi_{\mathcal{L}_L} = \{c\,\mu_L\}_{c\in\mathbb{R}^+}$, $\Xi_{\mathcal{L}_{(\alpha,\beta)}} = \{c\,\mu_{\alpha,\beta}\}_{c\in\mathbb{R}^+}$, where $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx, d\mu_L(x) = x^\alpha e^{-x}dx$ are the Jacobi and Laguerre measures, respectively. As a consequence, we obtain the following corollary,

COROLLARY 4.1. *Let* $\mathcal{L}$ *be a classical operator, i.e. Jacobi, Laguerre or Hermite and* μ *a positive Borel measure with support contained in* $\mathbb{R}$*. Then there exists an infinite sequence* $\{Q_n\}_{n=0}^\infty$ *of polynomials orthogonal with respect to* $(\mathcal{L},\mu)$*, with* $deg[Q_n] = n$ *if and only if* μ *is one of the measures* $c\,\mu_{\alpha,\beta}, c\,\mu_L, c\,\mu_H; c \in \mathbb{R}^+$*. In such case, all the sequences of monic orthogonal polynomials* $\{Q_n\}_{n=0}^\infty$ *with respect to the pair* $(\mathcal{L}^{(M)},\mu)$ *with* $deg[Q_n] = n$ *are of the form* $\{P_n + k_n\}_{n=0}^\infty$ *where* $\{k_n\}_{n=0}^\infty, k_0 = 0$*, is an arbitrary sequence of complex numbers and* $\{P_n\}_{n=0}^\infty$ *is the sequence of monic orthogonal polynomials with respect to* μ*.*

Proof. Let $\mathcal{L}$ be a fixed classical operator and $\{\lambda_n\}_{n=0}^\infty$ the associated sequence of eigenvalues. Then, we have that $\lambda_n = 0$ if and only if $n = 0$. Hence, the system (4.14) of Theorem 4.4 reduces to an unique equation. A simple calculation yields that the moments of the measure μ coincide with the moments of the measure of orthogonality of the sequence of eigenpolynomials of $\mathcal{L}$ multiplied by a real positive constant c (see Example 4.4.4 and the comment below it). Since the moment problem for a classical measure is determinate, we obtain that μ is the measure of orthogonality of the sequence of eigenpolynomials of $\mathcal{L}$ multiplied by a real positive constant c.

From Theorem 4.4 we have that for $n \geq 1$ there exists an infinite sequence $\{Q_n\}_{n\geq 1}$ of polynomials orthogonal with respect to $(\mathcal{L},\mu)$, with $deg[Q_n] = n$ if and only if μ is the measure of orthogonality of the sequence of eigenpolynomials of $\mathcal{L}$ multiplied by a real positive constant c. A simple calculation shows that for $n = 0$, the polynomial $Q_0 = 1$ satisfies the condition of orthogonality (1.9) and the statement is valid also for the sequence $\{Q_n\}_{n\geq 0}$.

It is not difficult to see that from the solutions of equations (4.2) and (4.3) we obtain that all the sequences of monic orthogonal polynomials $\{Q_n\}_{n\geq 0}$ with respect to the pair $(\mathcal{L}^{(M)},\mu)$ with $deg[Q_n] = n$ are of the form $\{P_n + k_n\}_{n=0}^\infty$ where $\{k_n\}_{n=0}^\infty, k_0 = 0$ is an arbitrary sequence of complex numbers and $\{P_n\}_{n=0}^\infty$ is the sequence of monic orthogonal polynomials with respect to μ. $\square$

Nevertheless, it is possible to guarantee the existence of a sequence $\{Q_n\}_{n>m}$, for some $m \in \mathbb{N}$ of polynomials orthogonal with respect to a classical operator for a measure μ which satisfies the condition $d\mu^*(x) = \rho(x)d\mu(x)$ where μ^* denotes the Jacobi, Hermite or Laguerre measure and ρ is a non negative polynomial on the support of μ^* of degree m, as will be shown

LEMMA 4.4. *Let* $\mathcal{L}$ *be a classical operator, i.e. Jacobi, Laguerre or Hermite,* μ *a finite positive Borel measure on* $\mathbb{R}$ *and* n *a fixed positive integer number. Then, differential equation* (4.2) *has an unique, except an additive constant, monic polynomial solution* Q_n *of degree* n *if and only if*

$$\int P_n(x)d\mu^*(x) = 0, \tag{4.17}$$

where P_n is the nth monic orthogonal polynomials with respect to the measure μ.

Proof. Suppose that there exists a polynomial Q_n of degree n such that $\mathcal{L}[Q_n] = \lambda_n P_n$. Let us denote by $\{L_n\}$ the sequence of orthogonal polynomials with respect to the measure μ^*. We have then

$$Q_n(z) = L_n(z) + \sum_{k=0}^{n-1} a_{(n,k)} L_k(z), \tag{4.18}$$

$$P_n(z) = L_n(z) + \sum_{k=0}^{n-1} b_{(n,k)} L_k(z), \tag{4.19}$$

where $a_{(n,k)} = \frac{\langle Q_n, L_k \rangle}{\langle L_k, L_k \rangle}$ and $b_{(n,k)} = \frac{\langle L_n, P_k \rangle}{\langle L_k, L_k \rangle}$.

Replacing Q_n and P_n in (4.2) by the linear combinations (4.18) and (4.19), from the linearity of $\mathcal{L}[\cdot]$ and the condition that $\mathcal{L}[L_n] = \lambda_n L_n$ we get

$$b_{(n,0)} = \frac{\int L_n(x)d\mu^*(x)}{\int d\mu^*} = 0.$$

Conversely, assume that P_n is the nth monic orthogonal polynomial with respect to μ fulfilling (4.17). Let Q_n the polynomial of degree n defined by

$$Q_n(z) = L_n(z) + \sum_{k=0}^{n-1} a_{(n,k)} L_k(z),$$

where $a_{(n,0)} = \Lambda_n$, where Λ_n is an arbitrary constant and $a_{(n,k)} = \frac{\lambda_n}{\lambda_k} \frac{\langle L_n, P_k \rangle}{\langle L_k, L_k \rangle}$. From the linearity of $\mathcal{L}[\cdot]$ and the condition that $\mathcal{L}[L_n] = \lambda_n L_n$ we get that $\mathcal{L}[Q_n] = \lambda_n P_n$. □

As a consequence, we have

THEOREM 4.5. *Let $\mathcal{L}$ be a classical operator, i.e. Jacobi, Laguerre or Hermite and μ be a finite positive Borel measure on $\mathbb{R}$, such that $d\mu^*(x) = \rho(x)d\mu(x)$, with $\rho \in L^2(\mu)$. Then, m is the smallest natural number such that for each $n > m$ there exists an infinite sequence $\{Q_n\}_{n>m}$ of polynomials orthogonal with respect to $(\mathcal{L}, \mu)$, with $deg[Q_n] = n$ if and only if ρ is a polynomial of degree m.*

Proof. Suppose that m is the smallest natural number such that for each $n > m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. According to Lemma 4.4

$$\int P_n(x)d\mu^*(x) = \int P_n(x)\rho(x)d\mu(x) \begin{cases} = 0 & \text{if } n > m, \\ \neq 0 & \text{if } n = m. \end{cases}$$

But this is equivalent to say that $\rho(x) = \sum_{k=0}^{m} c_k P_k(x)$ with $c_m \neq 0$. The converse is straightforward. □

Unlike Theorem 4.3, Theorem 4.4 does not guarantee the uniqueness of the sequence. A statement for the uniqueness can be obtained by fixing an adequate number of points in the complex plane. More precisely, we have

THEOREM 4.6. *Assume that $\mu \in \Xi_{\mathcal{L}^{(M)}}$ is not empty and let $\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(x) \frac{d^k}{dx^k}$ be an exactly solvable differential operator. Let the set S be as defined in Theorem 4.4 and let us fix (allowing repeated elements) $\{\nu_{1,n}, \ldots, \nu_{n_{j_0},n}\}$ points on the complex plane. Then, there exists an unique monic polynomial $R_{n-n_{j_0}}$ of degree $n - n_{j_0}$ such that*

$$Q_n(x) = (x - \nu_{1,n}) \cdots (x - \nu_{n_{j_0},n}) R_{n-n_{j_0}}(x),$$

is orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$. Here $n_{j_0} = dim\left(Ker[A_{n_{j_0+1}}]\right)$, where j_0 is the highest value for which $n_{j_0} < n$ and $n_{j_0} \in S$.

Proof. According to Theorem 4.4, if $\Xi_{\mathcal{L}^{(M)}}$ does not reduce to the empty set, then there exists a sequence $\{Q_n\}_{j \notin S}$ of monic polynomials orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$ and $deg[Q_n] = n$. Let $S = \{n_1, \ldots, n_k\}$.

Note that if $n < n_1$ then by $i)$ of Lemma 4.2 the index n is normal, hence the monic polynomial Q_n is unique. Assume that $n_1 < n$, and denote by

$$n_{j_0} = dim\left(Ker[A_{n_{j_0+1}}]\right),$$

where j_0 is the highest value for which $n_{j_0} < n$ and $n_{j_0} \in S$. Let us consider $\{Q_{n_j}\}_{j=1}^{n_{j_0}}$ a basis of monic polynomial solutions to (4.3) and assume that $\widehat{Q}_n$ is a monic polynomial solution of degree n to (4.2). Then, for a given index n, there exist unique coefficients $\{\alpha_j\}_{j=1}^{n_{j_0}}$ such that any monic polynomial solution Q_n of degree n to equation (4.2) can be expressed as

$$Q_n(x) = \widehat{Q}_n(x) + \sum_{j=1}^{n_{j_0}} \alpha_j Q_{n_j}(x). \tag{4.20}$$

Let us consider the multiset [18] $\{\nu_{1,n}, \ldots, \nu_{n_{j_0},n}\}$ of n_{j_0} points on the complex plane. We prove now the existence of a monic polynomial $R_{n-n_{j_0}}$ of degree $n - n_{j_0}$ such that

$$Q_n(x) = (x - \nu_{1,n}) \cdots (x - \nu_{n_{j_0},n}) R_{n-n_{j_0}}(x). \tag{4.21}$$

Later on, we shall address uniqueness. Assume that $\widehat{Q}_n$ does not vanish on $\{\nu_{1,n}, \ldots, \nu_{n_{j_0},n}\}$; otherwise, we have found our polynomial. Evaluating the polynomial

$$\widehat{Q}_n(x) + \alpha_1 Q_{n_1}(x) + \cdots + \alpha_{n_{j_0}} Q_{n_{j_0}}(x),$$

at the points $x = \nu_{j,n}$ and taking derivatives up to order $m_{\nu_{j,n}} - 1$, where $m_{\nu_{j,n}}$ is the number of times that the point $\nu_{j,n}$ appears in the multiset, we obtain that

$$\begin{aligned} \widehat{Q}_n(\nu_1) &= \alpha_1 Q_{n_1}(\nu_1) + \cdots + \alpha_{m_{j_0}} Q_{n_{j_0}}(\nu_1), & (4.22) \\ &\vdots \\ \widehat{Q}_n(\nu_{n_{j_0}}) &= \alpha_1 Q_{n_1}(\nu_1) + \cdots + \alpha_{n_{j_0}} Q_{n_{j_0}}(\nu_1). & (4.23) \end{aligned}$$

Defining $\alpha_1, \cdots \alpha_{n_{j_0}}$ as the solution of the above system, we obtain the existence.

We prove now that the polynomial $R_{n-n_{j_0}}$ is unique. Assume that there exist two different polynomials of degree n that vanish at the points $\{\nu_{1,n}, \ldots, \nu_{n_{j_0},n}\}$. Since the set $\{\widehat{Q}_n, Q_{n_j}, \ldots, Q_{n_{j_0}}\}$ is a basis for the polynomial solutions to (4.2), we have that

$$
\begin{aligned}
(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})R_{1,n-n_{j_0}}(x) &= \widehat{Q}_n(x)+\sum_{j=1}^{n_{j_0}}\alpha_{1,j}Q_{n_j},\\
(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})R_{2,n-n_{j_0}}(x) &= \widehat{Q}_n(x)+\sum_{j=1}^{n_{j_0}}\alpha_{2,j}Q_{n_j}.
\end{aligned}
$$

Both expressions give

$$
(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})(R_{1,n-n_{j_0}}(x)-R_{2,n-n_{j_0}}(x))=\sum_{j=1}^{n_{j_0}}(\alpha_{1,j}-\alpha_{2,j})Q_{n_j}.
$$

Note that the degree of the left hand side is strictly greater that n_{j_0} and the degree of the right hand side is at most n_{j_0}, which is a contradiction; that is, $R_{n-n_{j_0}}$ is unique. □

4.5 Zero location of the polynomials Q_n for a subclass of exactly solvable operators

In this section we study the location of the zeros of orthogonal polynomials with respect to a certain subclass of differential operators. We start with a discussion of the class of operators which we shall consider.

DEFINITION 4.1. *Given $M \geq 1$, we say that the linear differential operator $\mathcal{L}^{(M)}$ of M-th order factorizes on $\mathbb{P}$ if there exist multi-indexes $(m_1,\ldots,m_J)$, $(n_1,\ldots,n_J)$ and polynomials $\{H_{m_j}\}_{j=1}^{M}$ with $\deg[H_{m_k}]=m_k$, such that for each polynomial $\Pi_n\in\mathbb{P}$ we have*

$$
\mathcal{L}^{(M)}[\Pi_n](z)=\left[\rho_{m_J}(z)\cdots\left[\rho_{m_2}(z)\left[\rho_{m_1}(z)\Pi_n(z)\right]^{(n_1)}\right]^{(n_2)}\cdots\right]^{(n_J)}, \tag{4.24}
$$

for $k=1,\ldots,J$.

If $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$, we shall denote

$$
\begin{aligned}
\mathcal{L}_1^{(n_1)}[f](z) &:= (\rho_{m_1}(z)f(z))^{(n_1)},\\
&\vdots\\
\mathcal{L}_J^{(n_J)}[f](z) &:= (\rho_{m_J}(z)f(z))^{(n_J)},
\end{aligned}
$$

and then

$$
\mathcal{L}^{(M)}[f]=\mathcal{L}_J^{(n_J)}\circ\cdots\circ\mathcal{L}_1^{(n_1)}[f].
$$

We are interested in exactly solvable operators $\mathcal{L}^{(M)}$ which factorizes on $\mathbb{P}$, for the case in which ρ_{m_i} are polynomials with reals roots. According to Definition 1.8 of exactly solvable operator, we have necessarily that

$$
\sum_{k=1}^{J}m_k=\sum_{k=1}^{J}n_k=M. \tag{4.25}
$$

We denote by C_M the convex hull of the zeros of $\prod_{i=1}^{J} \rho_{m_i}$.

If $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$, then it is not difficult to see that $i)$ of Lemma 4.3 is equivalent to the condition,

$$\sum_{i=1}^{j}(m_i - n_i) \geq 0, \quad \forall j \leq J. \tag{4.26}$$

Hence, the class of operators that factorize on $\mathbb{P}$ for which there exists an unique infinite sequence $\{Q_n\}_{n=0}^{\infty}$, of monic polynomials, with $deg[Q_n] = n$, and orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$, for every positive Borel measure μ supported on $\mathbb{R}$, are those which satisfy condition (4.26).

To locate the zeros of orthogonal polynomials with respect to operators that factorize on $\mathbb{P}$ we use an integral representation for these operators and then we apply the known theorems for zero location of iterated integrals of polynomials. From the preceding discussions, it is already known that we have cases of operators for which the associated sequence of orthogonal polynomials is not unique. We will first analyze the class of operators defined by condition (4.26); that is, the class for which the existence of the full sequence $\{Q_n\}_{n=0}^{\infty}$ can be guaranteed. For these operators the following integral representation holds.

LEMMA 4.5. *Let P_n be the n-th monic orthogonal polynomial with respect to μ, $\mathcal{L}^{(M)}$ is such that factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)} = \mathcal{L}_J^{(n_J)} \circ \cdots \circ \mathcal{L}_1^{(n_1)}$ and satisfies* (4.26)*. Then, the following representation holds*

$$Q_n = \lambda_n I_1 \circ \cdots \circ I_J \left[P_n\right],$$

where I_j is the integration operator, given by

$$I_j[f](z) \quad = \quad \frac{1}{\rho_{m_j}(z)} \int_{z_{n_j,j}}^{z} \int_{z_{n_j-1,j}}^{t_{n_j-1}} \cdots \int_{z_{1,j}}^{t_1} f(t) dt dt_1 \cdots dt_{n_j-1},$$

and the sequence $\{z_{i,j}\} \subset C_M$.

Proof. As (4.26) holds, then by $ii)$ of Lemma 4.3 we have $\mathcal{L}^{(M)}[Q_n] = \lambda_n P_n$ is solvable. Let us consider the function

$$f(z) := \left[\rho_{m_J}(z) \mathcal{L}_{J-1}^{(n_{J-1})} \circ \cdots \circ \mathcal{L}_1^{(n_1)}[Q_n](z)\right]^{(n_J-1)}.$$

Applying successively Rolle's Theorem and taking into account that the polynomials ρ_{m_j} have their zeros on C_M we obtain that f has at least a zero $z_{1,J}$ in C_M. Hence, $f(z) = \lambda_n \int_{z_{1,J}}^{z} P_n(t) dt$.

By a similar argument we will have

$$\rho_{m_J}(z) \mathcal{L}_{J-1}^{(n_{J-1})} \circ \cdots \circ \mathcal{L}_1^{(n_1)}[Q_n](z) = \lambda_n \int_{z_{n_J,J}}^{z} \int_{z_{n_J-1,J}}^{t_{n_J-1}} \cdots \int_{z_{1,J}}^{t_1} P_n(t) dt dt_1 \cdots dt_{n_J-1}, \tag{4.27}$$

which implies that the polynomial $\int_{z_{n_J,J}}^{z} \int_{z_{n_J-1,J}}^{t_{n_J-1}} \cdots \int_{z_{1,J}}^{t_1} P_n(t) dt dt_1 \cdots dt_{n_J-1}$ is divisible by ρ_{m_J}. Therefore, after a finite number of steps we will have

$$\begin{aligned} Q_n(z) \quad &= \quad \lambda_n \frac{1}{\rho_{m_1}(z)} \int_{z_{n_1,1}}^{z} \int_{z_{n_1-1,1}}^{t_{n_1-1}} \cdots \int_{z_{1,1}}^{t_1} \cdots \frac{1}{\rho_{m_J}(z)} \\ &\left[\int_{z_{n_J,J}}^{z} \int_{z_{n_J-1,J}}^{t_{n_J-1}} \cdots \int_{z_{1,J}}^{t_1} P_n(t) dt dt_1 \cdots dt_{n_J-1}\right] \cdots dt dt_1 \cdots dt_{n_1-1} \\ &= \quad \lambda_n I_1 \circ \cdots \circ I_J \left[P_n\right](z). \end{aligned}$$

□

Consider now the class of operators which do not satisfy the condition (4.26). A representation similar to the one obtained in the preceding lemma can also be given. Let us prove some preliminary lemmas.

LEMMA 4.6. *Assume that $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)} = \mathcal{L}_J^{(n_J)} \circ \cdots \circ \mathcal{L}_1^{(n_1)}$. Then $Ker[\mathcal{L}^{(M)}] = \{0\}$ if and only if $\mathcal{L}^{(M)}[1] \neq 0$.*

Proof. The implication $Ker[\mathcal{L}^{(M)}] = \{0\} \quad \Rightarrow \quad \mathcal{L}^{(M)}[1] \neq 0$ is straightforward. Assume that $\mathcal{L}^{(M)}[1] \neq 0$. Note that

$$deg[\mathcal{L}_j^{(n_j)} \circ \cdots \circ \mathcal{L}_1^{(n_1)}[1]] = \sum_{i=1}^{j} (m_i - n_i),$$

from where we deduce that $\sum_{i=1}^{j} (m_i - n_i) \geq 0, \forall j \leq J$. Hence, from (4.26) and $iii)$ of Lemma 4.3 we obtain that $Ker[\mathcal{L}^{(M)}] = \{0\}$. □

LEMMA 4.7. *Assume that $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)} = \mathcal{L}_J^{(n_J)} \circ \cdots \circ \mathcal{L}_1^{(n_1)}$ and denote by j_0 the largest index such that $\sum_{i=1}^{j_0} (m_i - n_i) < 0$. Then $Ker[\mathcal{L}^{(M)}] = \{1, \ldots, x^{n_{j_0}}\}$, where $n_{j_0} = \sum_{i=1}^{j_0} (n_i - m_i) - 1$.*

Proof. Since $\mathcal{L}^{(M)}$ is a composition of operators, it is not difficult to see that if $1 \leq n \leq n_{j_0}$ then $\mathcal{L}^{(M)}[x^n] = 0$. Hence, $\{1, \cdots, x^{n_{j_0}}\} \subset Ker[\mathcal{L}^{(M)}]$.

Suppose now that $n = n_{j_0} + m, m \geq 1$. Then, we have

$$deg[\mathcal{L}_{j_0}^{(n_{j_0})} \circ \cdots \circ \mathcal{L}_1^{(n_1)}[x^n]] = m - 1 \geq 0,$$

and thus $\mathcal{L}^{(M)}[x^n] \neq 0$. □

An analogue of Lemma 4.5, for operators that do not satisfy condition (4.26), is

LEMMA 4.8. *Assume that $\mu \in \Xi_{\mathcal{L}^{(M)}} \neq \emptyset$, $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ as $\mathcal{L}^{(M)} = \mathcal{L}_J^{(n_J)} \circ \cdots \circ \mathcal{L}_1^{(n_1)}$ and suppose that condition (4.26) is not satisfied. Let us fix a multiset $\{\nu_{1,n}, \ldots, \nu_{n_{j_0},n}\}$ of real points, where n_{j_0} is as in Lemma 4.7, and let Q_n be the monic orthogonal polynomial with respect to $(\mathcal{L}^{(M)}, \mu)$ that vanish on $\{\nu_{1,n}, \ldots, \nu_{n_{j_0},n}\}$. Then, the following representation holds*

$$Q_n(x) = \lambda_n I_1 \circ \cdots I_{J-1} \circ \widehat{I}_{J,n} \left[P_n^{(n_{j_0})} \right](x), \quad n > n_{j_0},$$

where I_j is the integral operator given by

$$I_j[f](z) \quad = \quad \frac{1}{\rho_{m_j}(z)} \int_{z_{n_j},j}^{z} \int_{z_{n_j-1},j}^{t_{n_j-1}} \cdots \int_{z_{1,j}}^{t_1} f(t) dt dt_1 \cdots dt_{n_j-1},$$

$$\begin{aligned} \widehat{I}_{J,n}[f] &= I_J \circ I_{*,n}[f], \\ I_{*,n}[f] &= \int_{z^*_{n_{j_0}},J}^{z} \int_{z^*_{n_{j_0}-1},j}^{t_{n_{j_0}-1}} \cdots \int_{z^*_{1,j}}^{t_1} f(t) dt dt_1 \cdots dt_{n_{j_0}-1}, \end{aligned}$$

and the sequence $\{z_{i,j}, z^*_{i,j}\} \subset C^*_M$, where C^*_M is the convex hull of the zeros of

$$(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})\left(\prod_{i=1}^{J}\rho_{m_i}(x)\right).$$

Proof. By Lemma 4.7 we have that $Ker[\mathcal{L}^{(M)}] = \{1,\ldots,x^{n_{j_0}}\}$, where $n_{j_0} = \sum_{i=1}^{j_0}(n_i - m_i) - 1$. Set $S = \{0,\ldots,n_{j_0}\}$. Theorem 4.6 yields that there exists an unique monic polynomial $R_{n-n_{j_0}}$ such that if $n > n_{j_0}$

$$\mathcal{L}^{(M)}[\Pi_{n_{j_0}}R_{n-n_{j_0}}](x) = \lambda_n P_n(x),$$

where $\Pi_{n_{j_0}}(x) = (x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})$ and P_n is the n th monic orthogonal polynomial with respect to the measure μ. Taking derivatives up to order n_{j_0} in the above expression, we obtain

$$\mathcal{L}_J^{(n_J+n_{j_0})}\circ\cdots\circ\mathcal{L}_1^{(n_1)}[\Pi_{n_{j_0}}R_{n-n_{j_0}}](x) = \lambda_n P_n^{(n_{j_0})}(x)$$

or, equivalently,

$$\widehat{\mathcal{L}}[R_{n-n_{j_0}}](x) = \lambda_n P_n^{(n_{j_0})}(x),$$

where $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_J^{(n_J)}\circ\mathcal{L}_{J-1}^{(n_{J-1})}\circ\cdots\circ\mathcal{L}_2^{(n_2)}\circ\widehat{\mathcal{L}}_1^{(n_1)}$,

$$\begin{aligned}
\widehat{\mathcal{L}}_J^{(n_J)}[f] &= \mathcal{L}_J^{(n_J+n_{j_0})}[f], \quad f\in\mathbb{P},\\
\widehat{\mathcal{L}}_1^{(n_1)}[f] &= \widehat{\mathcal{L}}_1^{(n_1)}[\Pi_{n_{j_0}}f].
\end{aligned}$$

Since the polynomial $R_{n-n_{j_0}}$ is unique, we have that $\widehat{\mathcal{L}}[1]\neq 0$ and Lemma 4.6 gives that $Ker[\widehat{\mathcal{L}}] = \{0\}$. Therefore, from the equivalence of $iii)$ of Lemma 4.3 with (4.26), $\widehat{\mathcal{L}}$ satisfies $\sum_{i=1}^{j}(m_i-n_i)\geq 0, \forall j\leq J$. By Lemma 4.5, we obtain

$$R_{n-n_{j_0}}(x) = \lambda_n\widehat{I}_1\circ I_2\cdots I_{J-1}\circ\widehat{I}_J\left[P_n^{(n_{j_0})}\right](x)$$

or, equivalently,

$$Q_n(x) = \lambda_n I_1\circ\cdots I_{J-1}\circ\widehat{I}_J\left[P_n^{(n_{j_0})}\right](x), \quad n > n_{j_0},$$

where I_j is the integral operator given by

$$I_j[f](z) = \frac{1}{\rho_{m_j}(z)}\int_{z_{n_j,j}}^{z}\int_{z_{n_j-1,j}}^{t_{n_j-1}}\cdots\int_{z_{1,j}}^{t_1} f(t)dtdt_1\cdots dt_{n_j-1},$$

$$\begin{aligned}
\widehat{I}_1[f] &= \frac{1}{\Pi_{n_{j_0}}(x)}I_1[f], \quad f\in\mathbb{P},\\
\widehat{I}_{J,n}[f] &= I_J\circ I_{*,n}[f],\\
I_{*,n}[f] &= \int_{z^*_{n_{j_0}},J}^{z}\int_{z^*_{n_{j_0}-1},j}^{t_{n_{j_0}-1}}\cdots\int_{z^*_{1,j}}^{t_1} f(t)dtdt_1\cdots dt_{n_{j_0}-1},
\end{aligned}$$

and the sequence $\{z_{i,j}, z^*_{i,j}\} \subset C^*_M$, where C^*_M is the convex hull of the zeros of the polynomial

$$(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})\left(\prod_{i=1}^{J}\rho_{m_i}(x)\right).$$

□

4.5.1 Zero location

Assume that the exactly solvable operator $\mathcal{L}^{(M)}$ factorizes on $\mathbb{P}$ and that there exists an unique infinite sequence $\{Q_n\}_{n=0}^{\infty}$, of monic polynomials, each polynomial Q_n of degree equal to n, and orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$, for every positive Borel measure μ supported on $\mathbb{R}$. The following theorem, see [121, Exer. 20, pag. 74], and the results of the preceding section can be used now to locate the zeros of the family $\{Q_n\}$.

THEOREM 4.7. *If all the zeros of the nth degree polynomial f lie in a convex region K containing the point a, then all the zeros of $F(z) = \int_a^z f(t)dt$ lie in the domain bounded by the envelope of all circles passing through a and having centers on the boundary of K.*

The following lemma will be necessary for the zero location theorem.

LEMMA 4.9. *Let I_j be the integral operator defined in Lemma 4.5. Assume that the set $\{z_{i,j}\}_{i=1}^{n_j}$ and the zeros of the nth degree polynomial Π_n lie the circle $C(0,r)$ with center in the origin and radius r. Then the zeros of $I_j[\Pi_n]$ lie in the circle $C(0, 3^{n_j}r)$.*

Proof. If $z_{i,j}$ and the zeros of the nth degree polynomial Π_n lie in a circle $C(0,r)$ of radius r, by Theorem 4.7 the zeros of $\int_{z_{1,j}}^{z}\Pi_n(t)dt$ lie in the envelope of all the circles with center in the boundary of $C(0,r)$ and passing through $z_{1,j}$. It is not difficult to see that this envelope and the set $\{z_{i,j}\}_{i=2}^{n_j}$ are contained in the circle $C(0,3r)$. Using the same argument, we obtain that the zeros of $I_j[\Pi_n]$ are located in the circle $C(0,3^{n_j}r)$. □

Consider now the case of operators for which the full sequence of $\{Q_n\}_{n=0}^{\infty}$ exists, for every Borel measure μ supported on a subset of $\mathbb{R}$ or, equivalently, the operators for which this can be guaranteed are those which satisfy the condition (4.26). We have then,

THEOREM 4.8. *Let $\mathcal{L}^{(M)}$ be an exactly solvable operator that factorizes on $\mathbb{P}$ satisfying the condition* (4.26) *and μ a positive Borel measure supported in $[-1,1]$. Then, the zeros of the sequence $\{Q_n\}_{n=0}^{\infty}$, of monic polynomials orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$ are located in a circle of radius R, where $R = 3^M d$, with $d = \max\{1, \sup_{z\in C_M}|z|\}$.*

Proof. As $\{P_n\}_{n=0}^{\infty}$ is the sequence of orthogonal with respect to μ, their zeros are in $[-1,1]$. It is not difficult to see that the interval C_M and the zeros of the sequence $\{P_n\}_{n=0}^{\infty}$ are contained in a circle with center at the origin and radius $d = \max\{1, \sup_{z\in C_M}|z|\}$. From Lemma 4.5 we have that Q_n can be represented as

$$Q_n(z) = \lambda_n I_1 \circ \cdots \circ I_J\left[P_n\right](z).$$

Applying successively Lemma 4.9 we obtain that the zeros are located in a circle of radius R, where $R = 3^M d$, with $d = \max\{1, \sup_{z\in C_M}|z|\}$. □

Consider now the class of operators which do not satisfy the condition (4.26). In this case the associated sequence of orthogonal polynomials is not unique, nevertheless, in Theorem 4.6 it was shown that if we fix an adequate number of points we can define an unique infinite sequence of orthogonal polynomials. We have

THEOREM 4.9. *Let $\mathcal{L}^{(M)}$ be an exactly solvable operator that factorizes on $\mathbb{P}$ and assume that condition* (4.26) *is not satisfied, $\mu \in \Xi_{\mathcal{L}^{(M)}} \neq \emptyset$ such that $supp(\mu) \subset [-1,1]$ and consider a sequence of multisets $\{\nu_{1,n},\dots,\nu_{n_{j_0},n}\}$, where $\{\nu_{j,n}\} \subset \mathbb{R}$ and n_{j_0} are defined as in Lemma 4.7. Then, the zeros of the sequence $\{Q_n\}_{n=n_0+1}^{\infty}$ of monic orthogonal polynomials with respect to $(\mathcal{L}^{(M)},\mu)$ such that $Q_n(\nu_{j,n})=0, 1\le j\le n_{j_0}$ are located in a circle of radius R, where $R=3^M d$, with $d=\max\{1,\sup_{z\in C_M^*}|z|\}, C_M^*=\sup_n C_{M,n}^*$, and $C_{M,n}^*$ is the convex hull of the zeros of $(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})\left(\prod_{i=1}^{J}\rho_{m_i}(x)\right)$.*

Proof. By hypothesis, the zeros of the sequence $\{P_n\}_{n=0}^{\infty}$ are contained in $[-1,1]$. According to Theorem 4.6 there exists an unique sequence $\{Q_n\}_{n=n_0+1}^{\infty}$ of monic orthogonal polynomials with respect to $(\mathcal{L}^{(M)},\mu)$. By Lemma 4.8,

$$Q_n(x)=\lambda_n I_1\circ\cdots\circ\widehat{I}_J\left[P_n^{(n_{j_0})}\right](x),\quad n>n_{j_0},$$

where I_j is the integral operator given by

$$\begin{aligned}
I_j[f](z) &= \frac{1}{\rho_{m_j}(z)}\int_{z_{n_j,j}}^{z}\int_{z_{n_j-1,j}}^{t_{n_j-1}}\cdots\int_{z_{1,j}}^{t_1} f(t)dtdt_1\cdots dt_{n_j-1},\\
\widehat{I}_J[f] &= I_J\circ I_*[f],\\
I_*[f] &= \int_{z^*_{n_{j_0}},J}^{z}\int_{z^*_{n_{j_0}-1},j}^{t_{n_{j_0}-1}}\cdots\int_{z^*_{1,j}}^{t_1} f(t)dtdt_1\cdots dt_{n_{j_0}-1},
\end{aligned}$$

and the sequence $\{z_{i,j},z^*_{i,j}\}\subset C^*_{M,n}$, where $C^*_{M,n}$ is the convex hull of the zeros of the polynomial

$$(x-\nu_{1,n})\cdots(x-\nu_{n_{j_0},n})\left(\prod_{i=1}^{J}\rho_{m_i}(x)\right).$$

Note that the set $C^*_{M,n}$ and the zeros of $\{P_n\}$ are contained in a circle with center at the origin and radius $d=\max\{1,\sup_{z\in C^*_{M,n}}|z|\}$. Applying successively Lemma 4.9 we obtain that the zeros of Q_n are located in a circle of radius R_n, where $R_n=3^M d_n$, with $d_n=\max\{1,\sup_{z\in C^*_{M,n}}|z|\}$, from where we deduce that the zeros of the full sequence are located in a circle of radius $R=3^M d$, with $d=\max\{1,\sup_{z\in C^*_M}|z|\}, C^*_M=\sup_n C^*_{M,n}$. □

4.6 The polar polynomials case

In this section we study analytic properties of the polar polynomials, already introduced in Section 4.2. Let us denote by $d\mu_T(x)=\dfrac{1}{\sqrt{1-x^2}}dx$ the first kind Tchebychev measure and by T_n the n-th Tchebychev monic polynomial of the first kind. We shall study these polynomials for the class of finite positive Borel measures on $[-1,1]$ defined as $d\mu(x)=\dfrac{d\mu_T(x)}{\rho(x)}$ with $\rho(z)=r\prod_{i=1}^{m}(z-\nu_i)$ a non negative polynomial on $[-1,1]$. Denote by $\mathcal{P}_m(\mu_T)$ this class of measures. This complements the study carried out in [13] where the measure μ is the Gegenbauer measure. We obtain a curve which contains the accumulation points of the zeros of these polynomials and a formula for the strong asymptotic behavior of these polynomials in $\mathbb{C}\setminus[-1,1]$.

4.6.1 Strong asymptotic behavior and zero location

We recall that a measure supported on $[-1,1]$ is in the Szegö class $\mathfrak{S}$ if its absolutely continuous part μ' satisfies

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty.$$

The asymptotic properties of orthogonal polynomials with respect to a measure supported on $[-1,1]$ in the Szegö class can be described by means of the Szegö function $D(\mu, z)$, cf. [137, §6.1].

DEFINITION 4.2. *Let $\mu \in \mathfrak{S}$, then the Szegö function $D(d\mu, z)$ is defined by*

$$D(\mu(x), z) = \exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi} \log \mu'(\cos(t)) \frac{1+ze^{-\imath t}}{1-ze^{-\imath t}} dt\right]$$

for $|z|<1$.

It is well known that orthogonal polynomials with respect to a measure which belongs to the Szegö class has the following outer strong asymptotic behavior, cf. [137, §6.1 Lemma 18, page 67],

LEMMA 4.10. *Let μ be a positive Borel measure supported on $[-1,1]$, P_n the nth monic orthogonal polynomials associated to μ. Then*

$$\frac{\gamma_n P_n(z)}{\varphi(z)^n} \rightrightarrows \frac{1}{\sqrt{2\pi}} \left(D(\sqrt{1-x^2}d\mu(x), \varphi(z)^{-1})\right)^{-1}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$, where γ_n denotes the leading coefficient of the corresponding orthonormal polynomial of degree n.

The next lemmas are essential in the proof of the main theorem of this section

LEMMA 4.11. *Suppose that $\mu \in \mathcal{P}_m(\mu_T)$, then*

$$P_n(z) = \sum_{k=0}^{m} b_{n,n-k} T_{n-k}(z), \quad b_{n,n-k} = \frac{\int_{-1}^{1} P_n(x) T_{n-k}(x) d\mu_T(x)}{\int_{-1}^{1} T_{n-k}^2(x) d\mu_T(x)}, \tag{4.28}$$

where P_n, T_n are the monic orthogonal polynomials associated to the measures μ, μ_T, respectively, and the $b_{n-k,k}$ satisfy

$$\lim_{n\to\infty} b_{n,n-k} = 2^{m-k} a_k, \quad 0 \le k \le m, \tag{4.29}$$

where $a_k = (-1)^k \displaystyle\sum_{1\le \nu_1 < \cdots < \nu_k \le m} u_{\nu_1}^{-1} \dots u_{\nu_k}^{-1}, \quad u_{\nu_k} = \varphi(\nu_k).$

Proof. If $\mu \in \mathcal{P}_m(\mu_T)$ then $d\mu_T(x) = \rho(x) d\mu(x)$, where $\rho(x) = \displaystyle\prod_{i=1}^{m}(x-\nu_i)$ is nonnegative on $[-1,1]$. Therefore

$$P_n(z) = \sum_{k=0}^{m} b_{n,n-k} T_{n-k}(z), \quad b_{n,n-k} = \frac{\int_{-1}^{1} P_n(x) T_{n-k}(x) d\mu_T(x)}{\int_{-1}^{1} T_{n-k}^2(x) d\mu_T(x)}.$$

Hence, if $z = \frac{1}{2}(u+u^{-1})$ then $T_{n-k}(z) = \dfrac{u^{n-k}+u^{k-n}}{2^{n-k}}$, and

$$\frac{2^n P_n(z)}{u^n} = \sum_{k=0}^{m} 2^k b_{n,n-k} u^{-k} + \frac{1}{u^{2n}} \sum_{k=0}^{m} 2^k b_{n,n-k} u^k. \tag{4.30}$$

From [137, §6.1 theorem 26] and Definition 4.2, we have

$$\lim_{n\to\infty} \gamma_n 2^{-n} = \frac{1}{\sqrt{2\pi}} D(\rho(x), 0). \tag{4.31}$$

From Lemma 4.10 and (4.31), we obtain

$$\frac{2^n P_n(z)}{u^n} \rightrightarrows (D(\rho(x),0))^{-1} \left(D(\frac{1}{\rho(x)}, \varphi(z)^{-1}) \right)^{-1} = (D(\rho(x),0))^{-1} D(\rho(x), \varphi(z)^{-1}), \tag{4.32}$$

uniformly on closed subsets of $\mathbb{C} \setminus [-1,1]$. By [137, §6.1 Lemma 19] and Definition 4.2

$$\begin{aligned} D(\rho, \varphi(z)^{-1}) &= 2^m \exp\left(\frac{1}{2\pi} \int_{-1}^{1} \frac{\log(\rho(t))}{\sqrt{1-t^2}} dt \right) \prod_{k=1}^{m} \frac{z - \nu_k}{\varphi(z) - \varphi(\nu_k)}, \\ D(\rho, 0) &= \exp\left(\frac{1}{2\pi} \int_{-1}^{1} \frac{\log(\rho(t))}{\sqrt{1-t^2}} dt \right). \end{aligned}$$

Hence, if $z = \frac{1}{2}(u + u^{-1})$ the following identity holds

$$2^m \prod_{k=1}^{m} \frac{z - \nu_k}{\varphi(z) - \varphi(\nu_k)} = 2^m \prod_{k=1}^{m} \left(1 - \frac{1}{u u_{\nu_k}} \right) = \sum_{k=0}^{m} 2^m a_k u^{k-m}, \tag{4.33}$$

where $a_k = (-1)^k \sum_{1 \le \nu_1 < \cdots < \nu_k \le m} u_{\nu_1}^{-1} \ldots u_{\nu_k}^{-1}, u_{\nu_k} = \varphi(\nu_k)$.

From (4.30),(4.32), and (4.33),

$$\sum_{k=0}^{m} (2^k b_{n,n-k} - 2^m a_{m-k}) u^{-k} + \frac{1}{u^{2n}} \sum_{k=0}^{m} 2^k b_{n,n-k} u^k \rightrightarrows 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$. Therefore

$$\lim_{n\to\infty} b_{n,n-k} = 2^{m-k} a_k, \quad 0 \le k \le m.$$

□

LEMMA 4.12. *Suppose that $\mu \in \mathcal{P}_m(\mu_T)$. If K is a compact subset of $\overline{\mathbb{C}} \setminus [-1,1]$ and $\zeta \in \mathbb{C} \setminus [-1,1]$ then*

$$(z-\zeta) Q_n(z) = \frac{u^{n+1}}{2^{n+1}(n+1)} \Psi_n(u) - \frac{u_\zeta^{n+1}}{2^{n+1}(n+1)} \Psi_n(u_\zeta), \quad z = \frac{1}{2}(u + u^{-1}), \quad |u| > 1,$$

$$\Psi_n(u) \rightrightarrows \left(1 - \frac{1}{u^2} \right) (D(\rho,0))^{-1} D(\rho, u^{-1}), \quad u_\zeta = \varphi(\zeta).$$

Proof. From the definition of the polynomials Q_n, we have

$$(z-\zeta)Q_n(z) = \int_\zeta^z P_n(t)dt. \tag{4.34}$$

From (4.34) and (4.28), it follows that

$$(z-\zeta)Q_n(z) = \int_\zeta^z P_n(t)dt = \int_\zeta^z \sum_{k=0}^m b_{n,n-k}T_{n-k}(t)dt.$$

Making the change of variables $t = \dfrac{u+u^{-1}}{2}$, we obtain

$$(z-\zeta)Q_n(z) = \int_\zeta^z P_n(t)dt = \int_{\varphi(\zeta)}^{\varphi(z)} \sum_{k=0}^m b_{n,n-k}T_{n-k}\left(\frac{u+u^{-1}}{2}\right)\left(\frac{1}{2}-\frac{1}{2u^2}\right)du \tag{4.35}$$

Taking into account that for $n > m+1$

$$\int T_{n-k}\left(\frac{u+u^{-1}}{2}\right)\left(\frac{1}{2}-\frac{1}{2u^2}\right)du = \int \left(\frac{u^{n-k}+u^{-n+k}}{2^{n-k}}\right)\left(\frac{1}{2}-\frac{1}{2u^2}\right)du =$$

$$\frac{u^{n+1}}{2^{n+1}(n+1)}\left(g_{n-k}(u) - g_{n-2-k}(u)\right) + C, \tag{4.36}$$

where

$$g_{n-k}(u) = \frac{\frac{1}{2^{n-k}}\left(\frac{(n+1)}{(n-k+1)}u^{-k+n} + \frac{(n+1)}{(-n+k+1)}u^{k-n}\right)}{(\frac{u}{2})^n}.$$

Hence, if we denote

$$\Psi_n(u) = \sum_{k=0}^m b_{n,n-k}(g_{n-k}(u) - g_{n-2-k}(u)),$$

from (4.35) and (4.36), we obtain

$$(z-\zeta)Q_n(z) = \frac{u^{n+1}}{2^{n+1}(n+1)}\Psi_n(u)\Big|_{\varphi(\zeta)}^{\varphi(z)}.$$

From (4.32) of Lemma 4.11 and taking into account that $\lim_{n\to\infty} \dfrac{(n+1)}{(n-k-1)} = 1, 0 \le k \le m$, we obtain that

$$\Psi_n(u) \rightrightarrows \left(1-\frac{1}{u^2}\right)(D(\rho(x),0))^{-1}D(\rho(x),u^{-1}), \quad |u|>1.$$

□

THEOREM 4.10. *Suppose that $\mu \in \mathcal{P}_m(\mu_T)$, where μ_T is the first kind Tchebychev measure. Then the accumulation points of zeros of $\{Q_n\}_{n=0}^\infty$ are located on the set $E = \mathcal{E}(\zeta)\bigcup[-1,1]$, where $\mathcal{E}(\zeta)$ is the ellipse*

$$\mathcal{E}(\zeta) := \{z \in \mathbb{C} : z = \cosh(\eta_\zeta + i\theta), 0 \le \theta < 2\pi\}, \tag{4.37}$$

and $\eta_\zeta := \ln|\varphi(\zeta)| = \ln|\zeta + \sqrt{\zeta^2-1}|$. If $\delta(\zeta) > 2$ then $E = \mathcal{E}(\zeta)$.

Proof. From Lemma 4.12, the zeros of Q_n satisfy that

$$\left|\Psi_n(u)\frac{u^{n+1}}{2^{n+1}(n+1)}\right|^{\frac{1}{n}} = \left|\Psi_n(u_\zeta)\frac{u_\zeta^{n+1}}{2^{n+1}(n+1)}\right|^{\frac{1}{n}}, \quad z = \frac{1}{2}(u+u^{-1}), \quad |u|>1, \tag{4.38}$$

and from the definition of the function Ψ_n, we have that

$$\lim_{n\to\infty} |\Psi_n(u)|^{\frac{1}{n}} = 1, \quad |u|>1.$$

Therefore, taking limits on both sides of (4.38), we deduce that the zeros of Q_n can not accumulate outside the set

$$\left\{z\in\mathbb{C} : |z+\sqrt{z^2-1}| = e^{\eta_\zeta}\right\} \bigcup [-1,1] \bigcup \{\zeta\}.$$

Hence, if z is an accumulation point of zeros of the polynomials Q_n, we have that $z+\sqrt{z^2-1} = e^{\eta_\zeta+i\theta}$ and $z-\sqrt{z^2-1} = e^{-(\eta_\zeta+i\theta)}$ for $0\le\theta<2\pi$, and $2z = e^{\eta_\zeta+i\theta}+e^{-(\eta_\zeta+i\theta)}$. $\square$

Chapter 5

Strong asymptotic behavior of the eigenpolynomials of exactly solvable operators.

5.1 Introduction

In this chapter we consider a M–th order differential linear operator

$$\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k},$$

where ρ_M is a monic complex polynomial such that $deg[\rho_M] = M$ and $\{\rho_k\}_{k=0}^{M-1}$ are complex polynomials such that $deg[\rho_k] \leq k, 0 \leq k \leq M-1$. Under the assumption that ρ_M is real, we obtain a formula for the strong asymptotic behavior of the eigenpolynomials of $\mathcal{L}^{(M)}$ on certain compact subsets of $\mathbb{C}$. As an application we obtain an asymptotic formula for the sequence $\{Q_n\}_{k=0}^{\infty}$ of monic orthogonal polynomials with respect to the Sobolev inner product,

$$\langle P, Q \rangle = P(1)\overline{Q}(1) + c\, P'(1)\overline{Q}'(1) + \int_{-1}^{1} P'\overline{Q}'\, dx, \quad P, Q \in \mathbb{P}, \quad c > 0$$

for compact subsets of $\mathbb{C} \setminus [-1, 1]$, where $\mathbb{P}$ denotes the vector space of all polynomials over $\mathbb{C}$.

Let $M \geq 2$ be an integer and let $\{\rho_0, \rho_1, \ldots, \rho_M\}$ be a set of $(M+1)$ polynomials in one complex variable such that $deg[\rho_k] \leq k$ for $k = 0, 1, \ldots, M$ and at least one of them, say ρ_{k^*}, is exactly of degree k^*. Consider the linear ordinary differential operator of order M

$$\mathcal{L}^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k}, \tag{5.1}$$

where $\rho_k(x) = \sum_{j=0}^{k} \rho_{k,j} x^j$ and without loss of generality, assume that ρ_M is a monic polynomial. Note that if f_n is a polynomial of degree n then $\mathcal{L}^{(M)}[f_n]$ is a polynomial of the same degree. Linear differential operators that satisfy the previous property of invariance are called *exactly–solvable* (cf. [166]). They split in two classes: non degenerate, if the leading term of the operator satisfies $[\rho_M] = M$ and degenerate, if $deg[\rho_M] < M$. Bochner–Krall operators are a particular case of this class [99].

Some analytic and algebraic properties of the eigenpolynomials of exactly solvable operators of the form

$$\mathcal{L}^{(M)}[f](z) = \frac{d^M}{dz^M}\left(\rho_M(z) f(z)\right),$$

where ρ_M is a fixed polynomial of degree M have been previously studied in [132], and for exactly solvable operators, in general, by [15], [16], [17].

Given a polynomial P_n of degree n, denote $\nu_n = \dfrac{1}{n}\displaystyle\sum_{P_n(z_k)=0} \delta_{z_k}$. If the following limit exists (in the sense of the weak convergence) $\nu = \lim_{n\to\infty} \nu_n$, then ν is called the asymptotic zero-counting measure of the sequence $\{P_n\}_{n=0}^{\infty}$. It is known that for n large enough the operator (5.1) has a unique eigenpolynomial and that the zero counting measures of the eigenpolynomials converge weakly to a measure ν, with support contained in the convex hull of the zeros of ρ_M, moreover, this support is connected and its complement is also connected, cf. [15, Ths. 3,4].

The main purpose of this chapter is the proof of a formula for the strong asymptotic behavior of the monic polynomial eigenfunctions Q_n of $\mathcal{L}^{(M)}$ on compact subsets of $\mathbb{C}\setminus supp(\nu)$. Formulas of this type have drawn a great deal of attention in connection with problems of the theory of orthogonal polynomials and approximation theory.

According to [15], for n sufficiently large there exists a unique polynomial eigenfunction Q_n for the linear exactly solvable differential operator (5.1) with eigenvalue given by,

$$\lambda_n = \sum_{0\le k\le \min(M,n)} \rho_{k,k}\frac{n!}{(n-k)!}.$$

Consider the differential equation

$$\mathcal{L}^{(M)}[u](z) - \frac{u(z)}{\varepsilon^M} = 0, \quad \varepsilon \in \mathbb{C}\setminus\{0\},$$

or, equivalently,

$$u^{(M)}(z) + \sum_{k=1}^{M-1} \frac{\rho_k(z)}{\rho_M(z)} u^{(k)}(z) - \frac{u(z)}{\varepsilon^M \rho_M(z)} \quad = \quad 0. \tag{5.2}$$

It is not difficult to see that Q_n is a polynomial solution of (5.2) with $\varepsilon_n^M = (\lambda_n - \rho_{0,0})^{-1}$, where $\rho_{0,0} = \rho_0(z)$. Conversely, if for some quantity ε_n, u is a polynomial solution of (5.2) with $deg[u] = n$, then u is a polynomial eigenfunction of the exactly solvable differential operator (5.1) with eigenvalue $\lambda_n = \rho_{0,0} + \varepsilon_n^{-M}$.

In order to study the asymptotic behavior of the sequence of eigenpolynomial $\{Q_n\}_{n=0}^{\infty}$, we will find M linearly independent solutions for (5.2) of the form $e^{y_j(z,\varepsilon_n)}$, $j = 1,\dots,M$, where $y_j(z,\varepsilon)$ is expressed by a convergent infinite series of the form $\displaystyle\sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}$ and for each j,k fixed $b_{j,k}$ is holomorphic on the exterior of any closed Jordan curve containing $supp(\nu)$. Functions of this form are known as WKB solutions for a given differential equation and are widely used in asymptotic expansions of solutions of differential equations cf. [64, 141]. In general, it is not known if this kind of solutions exists. As far we know, there is no proof of the convergence of the series that define the WKB solutions for differential equations of the form (5.2).

As usual, we say that $U \subset \mathbb{C}$ is a *domain* if U is a connected open set. The set $\mathcal{H}(U)$ will denote the space of all single-valued analytic functions on U and $\mathcal{L}^2(U)$, the space of all square integrable functions on U with respect to the area measure.

Let γ be a closed Jordan curve satisfying $supp(\nu) \subset int(\gamma)$. According to the Jordan curve's Theorem, γ divides the complex plane in two disjoint regions, we denote by $int(\gamma)$ the region which does not contain the ∞ point and by $ext(\gamma)$ the region which contains the ∞ point. Assume that $\zeta \in \gamma$ and that $\mathfrak{r}_\zeta$ is a Jordan curve from ζ to the point at infinity such that $\mathfrak{r}_\zeta \bigcap \gamma = \{\zeta\}$ then we define the simply connected open set $G_{\gamma,\mathfrak{r}_\zeta} = ext(\gamma) \setminus \mathfrak{r}_\zeta$. Denote by V_0^* a reduced neighborhood of $\varepsilon = 0$. We prove then that,

THEOREM 5.1. *Let $\mathcal{L}^{(M)}$ be a non degenerate exactly solvable operator and let γ be a closed Jordan curve such that $supp(\nu) \subset int(\gamma)$, then there exists M linearly independent holomorphic eigenfunctions of the form $u_j(z,\varepsilon) = e^{y_j(z,\varepsilon)}$ for* (5.2) *on $G_{\gamma,\mathfrak{r}_\zeta} \times V_0^*$, where $y_j(z,\varepsilon) = \sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}$ and for each j,k fixed $b_{j,k} \in \mathcal{H}(G_{\gamma,\mathfrak{r}_\zeta})$.*

Now assume that the leading coefficient ρ_M of the operator $\mathcal{L}^{(M)}$ is a real polynomial. Take $\zeta = \min_{z\in\gamma\bigcap\mathbb{R}} z$ and $\mathfrak{r}_\zeta = (-\infty,\zeta)$.

Let us denote by Φ_0 the primitive on $G_{\gamma,\mathfrak{r}_\zeta}$ of the branch of the function $\dfrac{1}{\sqrt[M]{\rho_M(z)}}$ which coincides with $\dfrac{1}{z}$ at ∞ such that $\lim\limits_{\substack{z\to\infty\\ z\in G_{\gamma,\mathfrak{r}_\zeta}}} \Phi_0(z) - \ln z = 0$. Define Φ_0 on $z_0 \in (-\infty,\zeta)$ as $\lim\limits_{\substack{z\to z_0\\ \Im(z)>0}} \Phi_0(z)$.

In a similar way, Φ_1 denotes the primitive on $G_{\gamma,\mathfrak{r}_\zeta}$ of the function

$$\frac{(M-1)\rho_M'(z)}{2M\rho_M(z)} - \frac{\rho_{M-1}(z)}{M\rho_M(z)},$$

such that $\lim\limits_{\substack{z\to\infty\\ z\in G_{\gamma,\mathfrak{r}_\zeta}}} \Phi_1(z) - \left(\dfrac{M-1}{2} - \dfrac{\rho_{M-1,M-1}}{M}\right)\ln z = 0$. Define Φ_1 on $z_0 \in (-\infty,\zeta)$ as $\lim\limits_{\substack{z\to z_0\\ \Im(z)>0}} \Phi_1(z)$.

As a consequence of Theorem 5.1, we deduce a formula for the strong asymptotic behavior of the sequence of eigenpolynomials $\{Q_n\}_{n=0}^\infty$,

THEOREM 5.2. *Let $\mathcal{L}^{(M)}$ be a non degenerate exactly solvable operator such that ρ_M is a real polynomial, let Q_n be the n–th monic eigenpolynomial of $\mathcal{L}^{(M)}$ and γ a closed Jordan curve such that $supp(\nu) \subset int(\gamma)$, then*

a) $Q_n(z) = e^{\left(n - \left(\frac{M-1}{2} - \frac{\rho_{M-1,M-1}}{M}\right)\right)\Phi_0(z) + \Phi_1(z)}(1+O(1/n))$

b) $\lim\limits_{n\to\infty} \dfrac{Q_{n+1}(z)}{Q_n(z)} = e^{\Phi_0(z)}$

c) $\lim\limits_{n\to\infty} \sqrt[n]{|Q_n(z)|} = e^{\operatorname{Re}[\Phi_0(z)]}$,

uniformly on compacts subsets $K \subset ext(\gamma)$.

Consider the sequence of monic orthogonal polynomials with respect to the Sobolev inner product,

$$\langle P,Q\rangle = P(1)\overline{Q}(1) + \frac{1}{c}P'(1)\overline{Q}'(1) + \int_{-1}^{1} P'\overline{Q}'dx, \quad P,Q\in\mathbb{P}, \quad c>0,$$

which are eigenfunctions of the fourth order differential operator, cf.[89],

$$\mathcal{L}^{(M)}[u] = (z^2-1)^2u^{(4)} + 4z(z^2-1)u^{(3)} + 2(z-1)((1+2c)z+2c+3)u^{(2)}. \tag{5.3}$$

Using Theorem 5.2 we obtain the following asymptotic formula,

$$\lim_{n\to\infty} Q_n(z) = \left(\frac{\varphi(z)}{2}\right)^{n-1/2}\sqrt[4]{z^2-1}\,(1+O(1/n)), \quad z\in K\subset\backslash\mathbb{C}, \tag{5.4}$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$ and we take the branch of $\sqrt{z^2 - 1}$ for which $|\varphi(z)| > 1$ whenever $z \in \mathbb{C} \setminus [-1, 1]$. Here we choose the branch of the roots $\sqrt[4]{z}$, $\sqrt{z}$ from the conditions $\sqrt[4]{1} = 1$, $\sqrt{1} = 1$.

The chapter is organized as follows. The proof of Theorems 5.1 and 5.2 is carried out in Sections 5.2 and 5.3 respectively. In Section 5.4 we deduce Formula (5.4) for the strong asymptotic behavior of the orthogonal polynomials with respect to the Sobolev inner product given by (5.3). The contents of this chapter have been submitted for consider for publication, see [25].

5.2 Functional spaces, multilinear analytic operators and the convergence of exponential series.

In this section we prove the existence of M linearly independent solutions for the differential equation (5.2) in the form of an exponential convergent series of holomorphic functions. These expressions shall be used in the next section for the proof of the strong asymptotic behavior of the sequence of eigenpolynomials of (5.1). We begin with some notation and background on Bergman–Sobolev spaces and analytic operators used in the proofs of some lemmas.

5.2.1 Bergman–Sobolev spaces and analytic operators.

Given a bounded domain in the complex plane U and a positive number p, the *Bergman space with exponent p for the domain* consists of the analytic functions on the domain, whose modulus is pth power integrable with respect to area (cf. [60, 86]). For $p = 2$ the Bergman space is denoted by $\mathcal{A}^2(U)$, i.e. the space of all single-valued analytic functions f defined in the domain U such that

$$\|f\|_{\mathcal{A}^2(U)} = \sqrt{\iint_U |f(z)|^2 dA} < \infty, \quad z = x + iy,$$

where all integrals are understood in the Lebesgue sense and $dA = dxdy$ denotes the planar Lebesgue measure on U. Let $\mathcal{H}(U)$ be the space of all single-valued analytic functions on U and $\mathcal{L}^2(U)$ be the space of all square integrable functions on U with respect to the area measure, then $\mathcal{A}^2(U) = \mathcal{H}(U) \bigcap \mathcal{L}^2(U)$.

As is well known (cf. [60, §1.1] and [94, Lem. 1.4.2]), $\mathcal{A}^2(U)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{\mathcal{A}^2(U)} = \iint_U f(z)\overline{g(z)}\, dA,$$

$\overline{g}$ denotes the complex conjugate of g.

In the proof of Lemma 5.1, we need the so called *Bergman–Sobolev space of order M for the domain U* defined as

$$\mathcal{A}^{2,M}(U) = \{f \in \mathcal{A}^2(U) : f^{(j)} \in \mathcal{A}^2(U) \text{ for all } 1 \leq j \leq M\},$$

with inner product given by

$$\langle f, g \rangle_{\mathcal{A}^{2,M}(U)} = \sum_{j=0}^{M} \iint_U f^{(j)}(x + \imath y))\overline{g^{(j)}(x + \imath y)}\, dxdy. \tag{5.5}$$

PROPOSITION 5.2.1. *Let U be a bounded domain of the complex plane, then $(\mathcal{A}^{2,M}(U), \langle\cdot\rangle_{\mathcal{A}^{2,M}(U)})$ is a separable Hilbert space.*

Proof. From the inclusions $\mathcal{A}^{2,M}(U) \subset \mathcal{A}^2(U) \subset \mathcal{L}^2(U)$ we deduce that $\mathcal{A}^{2,M}(U)$ is a separable vector space.

Let's prove now that $\mathcal{A}^{2,M}(U)$ is complete. Let $\{f_n\} \subset \mathcal{A}^{2,M}(U)$ be a Cauchy sequence on $\mathcal{A}^{2,M}(U)$, then $f_n, f_n^{(1)}, \ldots, f_n^{(M)}$ are Cauchy sequences on the Hilbert space $\mathcal{A}^2(U)$ which implies that there exists $f, f_1, \ldots, f_M \in \mathcal{A}^2(U)$ such that $\lim\limits_{n\to\infty} \|f_n - f\|_{\mathcal{A}^2(U)} \to 0$ and $\lim\limits_{n\to\infty} \|f_n^{(j)} - f_j\|_{\mathcal{A}^2(U)} \to 0, 1 \leq j \leq M$, then from [122, Th. 3.21, Vol III] we deduce that $f_n \to f$ and $f_n^{(j)} \to f_j, 0 \leq j \leq M$ uniformly on $int(U)$, hence $f^{(j)} = f_j, 1 \leq j \leq M$ from where we deduce the completeness of $\mathcal{A}^{2,M}(U)$. $\square$

Suppose that X and Y are Banach spaces over the field $\mathbb{C}$, $k \in \mathbb{N}$ and $X^k = X \times \cdots \times X$, k times. A mapping $\Psi : X^k \longrightarrow Y$ is said to be a *multilinear operator* (k-linear in this case) if it is linear in each variable separately. It is said to be *bounded* (or *continuous*, as in the linear case) multilinear operator if, in addition, $\sup_{\substack{\|x_i\|\leq 1\\ i=1,\ldots,k}} \|\Psi(x_1, x_2, \ldots, x_k)\| < \infty$. Let $\mathcal{M}^k(X, Y)$ be the set of all bounded multilinear operators endowed with the norm

$$|||\Psi||| = \sup_{\substack{\|x_i\|\leq 1\\ i=1,\ldots,k}} \|\Psi(x_1, x_2, \ldots, x_k)\|.$$

As it is known, $(\mathcal{M}^k, |||\cdot|||)$ is a Banach space.

An operator $\Psi \in \mathcal{M}^k(X, Y)$ is called *symmetric* if $\Psi(x_1, x_2, \ldots, x_k) = \Psi(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$, where $\pi \in \mathcal{S}_k$ and $\mathcal{S}_k$ is the group of permutations of the set $\{1, 2, \ldots, k\}$. The symmetric operators form a closed subspace of $\mathcal{M}^k(X, Y)$.

The higher order Fréchet derivatives of a function $F : U \subset X \longrightarrow Y$, where U is an open subset, is a standard example of bounded symmetric multilinear operator [29, Prop. 3.2.3].

A mapping $F : U \longrightarrow Y$ is *analytic* at $x_0 \in U$ if, for all $x \in U$ with $\|x - x_0\|$ sufficiently small

$$F(x) = \sum_{k=0}^{\infty} f_k(x - x_0)^k, \tag{5.6}$$

where $F(x_0) = f_0(x - x_0)^0 = f_0 \in Y$ and $f_k(x - x_0)^k \equiv f_k(\underbrace{x - x_0, \ldots, x - x_0}_{\text{k times}})$, $f_k \in \mathcal{M}^k(X, Y)$ is symmetric and there exist $r > 0$ such that $\sup_{k\geq 0} r^k |||f_k||| = M < \infty$.

The mapping F is said to be analytic in U if it is analytic at every point of U.

5.2.2 Linearly independent solutions.

Let us denote by $P_k(y^{(1)}, \ldots, y^{(k)})$ the polynomial in the functional variables $(y^{(1)}, \ldots, y^{(k)})$ defined for all integers $1 \leq k \leq M$ by the relation

$$P_k(y^{(1)}, \ldots, y^{(k)}) = e^{-y}\ (e^y)^{(k)}.$$

According to the Faa di Bruno formula [150], $P_k(y^{(1)}, \ldots, y^{(k)})$ can be expressed as

$$P_k(y^{(1)}, \ldots, y^{(k)}) = \sum \binom{k}{n_1, \ldots, n_k} \left(\frac{y^{(1)}}{1!}\right)^{n_1} \cdots \left(\frac{y^{(k)}}{k!}\right)^{n_k}, \tag{5.7}$$

if $k < N$, where $\displaystyle\binom{k}{n_1, \ldots, n_k} = \frac{k!}{n_1! \cdots n_k!}$, and the sum is over all partitions of k, i.e., values of $n_1, \ldots, n_k \in \mathbb{N}$ such that $n_1 + 2n_2 + \cdots + kn_k = k$.

For convenience, let additionally

$$\begin{aligned}\widehat{P}_M(y^{(1)},\ldots,y^{(M-1)}) &= e^{-y}\,(e^y)^{(M)} - y^{(M)} - \left(y^{(1)}\right)^M \\ &= P_M(y^{(1)},\ldots,y^{(M-1)}) - y^{(M)} - \left(y^{(1)}\right)^M.\end{aligned} \tag{5.8}$$

In the sequel $V_0 \subset \mathbb{C}$ will denote a neighborhood of 0 and $V_0^* = V_0 \setminus \{0\}$. Let γ be a closed Jordan curve such that $supp(\nu) \subset int(\gamma)$. Assume that $\zeta \in \gamma$ and that $\mathfrak{r}_\zeta$ is a Jordan curve from ζ to the point at infinity such that $\mathfrak{r}_\zeta \bigcap \gamma = \{\zeta\}$. The following Lemma proves the existence of solutions in the form of exponential series for some open sets of the complex plane.

LEMMA 5.1. *Assume that ρ_M in the differential equation* (5.2) *satisfies $deg[\rho_M] = M$ and let γ be a closed Jordan curve and C a circle such that $supp(\nu) \subset int(\gamma) \subsetneq int(C)$, $U = \left[G_{\gamma,\mathfrak{r}_\zeta} \setminus ext(C)\right] \setminus C_1$ and $\{z_{j,k}\}_{\substack{j=1,\ldots,M\\ k=0,\ldots,\infty}} \subset U$ a sequence of complex numbers. Then, there exist M linearly independent holomorphic solutions of the differential equation* (5.2) *of the form*

$$u_j(z,\varepsilon) = e^{y_j(z,\varepsilon)}, \quad (z,\varepsilon) \in U \times V_0^*, \quad j = 1,\ldots,M, \tag{5.9}$$

where

$$y_j(z,\varepsilon) = \sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}, \tag{5.10}$$

and for each j fixed, $1 \le j \le M$, $\{b_{j,k}\}_{k=0}^{\infty}$ is a unique sequence of single valued analytic functions on U such that $b_{j,k}(z_{j,k}) = 0$.

Proof. The relations (5.7),(5.8) and the transformation of the variable $u = e^y$ give us that (5.2) can be expressed as

$$y^{(M)} + \left(y^{(1)}\right)^M + \widehat{P}_M(y^{(1)},\ldots,y^{(M-1)}) + \sum_{k=1}^{M-1} \frac{\rho_k(z)}{\rho_M(z)} P_k(y^{(1)},\ldots,y^{(k)}) - \frac{1}{\rho_M(z)\varepsilon^M} = 0. \tag{5.11}$$

Multiplying (5.11) by ε^M, transforming the variable

$$y^{(1)} = w\varepsilon^{-1}, \tag{5.12}$$

and using the homogeneity of P_k we obtain,

$$w^M + \varepsilon^{M-1}\left(w^{(M-1)} + \widehat{P}_M(w,\ldots,w^{(M-2)}) + \sum_{k=1}^{M-1} \frac{\rho_k(z)}{\rho_M(z)} P_k(w,\ldots,w^{(k-1)})\right) - \frac{1}{\rho_M(z)} \quad = \quad 0.$$

Let $\mathcal{D} \subset \mathbb{C}$ be the unit disk and $F(w,\varepsilon)$ the differential expression

$$\begin{aligned}F(w,\varepsilon) &= w^M + \varepsilon^{M-1}w^{(M-1)} + \varepsilon^{M-1}\widehat{P}_M(w,w^{(1)},\ldots,w^{(M-2)}) \\ &\quad + \varepsilon^{M-1}\sum_{k=1}^{M-1} \frac{\rho_k(z)}{\rho_M(z)} P_k(w,w^{(1)},\ldots,w^{(M-2)}) - \frac{1}{\rho_M(z)}.\end{aligned} \tag{5.13}$$

Now consider F as the operator

$$\begin{aligned} F : \mathcal{A}^{2,M}(U) \times \mathcal{D} &\longrightarrow \mathcal{A}^{2,M}(U) \\ (w,\varepsilon) &\longrightarrow F(w,\varepsilon). \end{aligned}$$

Notice that F is an analytic operator ($\mathbb{F}$-analytic, where $\mathbb{F}$ denotes the field of scalars, $\mathbb{R}$ or $\mathbb{C}$) in the sense of [29, Def. 4.3.1 and §4.4]. In other words, let $\mathcal{U} \subset \big(\mathcal{A}^{2,M}(U) \times \mathcal{D}\big)$ be an open subset and $(w_0, \varepsilon_0) \in \mathcal{U}$, for (w,ε) sufficiently close to (w_0,ε_0)

$$F(w,\varepsilon) = \sum_{k=0}^{\infty} f_k(w - w_0, \varepsilon - \varepsilon_0)^k,$$

where for each k

$$\begin{aligned} f_k : \underbrace{(\mathcal{A}^{2,M}(U) \times \mathcal{D}) \times \cdots \times (\mathcal{A}^{2,M}(U) \times \mathcal{D})}_{k \text{ times}} &\rightarrow \mathcal{A}^{2,M}(U) \\ ((w,\varepsilon),\ldots,(w,\varepsilon)) &\rightarrow f_k(w,\varepsilon)^k = f_k((w,\varepsilon),\ldots,(w,\varepsilon)) \end{aligned}$$

is a multilinear, symmetric and bounded operator (cf. [29, Ch. 4]). From (5.13) we deduce that for the operator F only a finite number of operators f_k are different from the null one and are determined by each one of the summands of (5.13).

It is not difficult to see that for the function $\left(\sqrt[M]{\rho_M(z)}\right)^{-1}$, $z \in U$, it is possible to separate M single valued analytic branches on U. Let us define $\widehat{b}_{j,0,U} \equiv \widehat{b}_{j,0}(z) = \left(\sqrt[M]{\rho_M(z)}\right)^{-1}$, $z \in U$, where the index $j = 1,\ldots,M$ denotes each branch of the root. Notice that $\widehat{b}_{j,0} \in \mathcal{H}(U)$ and $F(\widehat{b}_{j,0}, 0) = 0$, for every j.

Let us fix an index j and denote by $\partial_w F[(\widehat{b}_{j,0},0)]$ the *partial Fréchet derivative* of F with respect to w at $(\widehat{b}_{j,0},0)$ (cf. [29, Def. 3.1.5]). From (5.13), $\partial_w F[(\widehat{b}_{j,0},0)]$ is the bounded linear operator

$$\begin{aligned} \partial_w F[(\widehat{b}_{j,0},0)] : A^{2,M}(U) &\longrightarrow A^{2,M}(U), \\ h &\longrightarrow M\widehat{b}_{j,0}^{M-1} h, \end{aligned}$$

which is an homeomorphism from $A^{2,M}(U)$ to itself.

From the *analytic implicit function theorem in Banach spaces* (cf. [29, Th. 4.5.4]) with $\varepsilon_0 = 0$, there exists a neighborhood of 0, $V_0 \subset \mathcal{D}$, and an unique analytic function $\phi_j : V_0 \to A^{2,M}(U)$, such that

$$\begin{aligned} &F(\phi_j(\varepsilon),\varepsilon) = 0, \\ &\phi_j(0) = \widehat{b}_{j,0}(z). \end{aligned} \tag{5.14}$$

As ϕ_j is analytic, $\phi_j(\varepsilon) = \sum_{k=0}^{\infty} \eta_{j,k}(\varepsilon)^k$, where for each k

$$\begin{aligned} \eta_k : \underbrace{V_0 \times \cdots \times V_0}_{\text{k times}} &\longrightarrow A^{2,M}(U) \\ (\varepsilon,\ldots,\varepsilon) &\longrightarrow \eta_{j,k}(\varepsilon)^k = \eta_{j,k}(\varepsilon,\ldots,\varepsilon), \end{aligned}$$

is a multilinear symmetric bounded operator.

From Proposition 5.2.1 we have that $A^{2,M}(U)$ is a separable Hilbert space, hence its dual is separable, and from the integral representation Theorem for multilinear bounded operators in Banach spaces [20, Th. 3], we deduce that $\eta_{j,k}(\varepsilon)^k = \widehat{b}_{j,k}(z)\varepsilon^k$, where for each $0 \le k \le \infty$, $\widehat{b}_{j,k} \in \mathcal{H}(U)$. Let us define $w_j(z,\varepsilon) = \phi_j(\varepsilon)$.

From the above discussion we have then that there exist M linearly independent functions of the form $w_j(z,\varepsilon) = \sum_{k=0}^{\infty} \widehat{b}_{j,k}(z)\varepsilon^k$, where $\{\widehat{b}_{j,k}\}_{k=0}^{\infty}$ are analytic on U and the index j denotes a fixed branch of the function $w_0(z) = \dfrac{1}{\sqrt[M]{\rho_M(z)}}$. Applying the inverse transformation (5.12) we obtain that $y_j(z,\varepsilon) = \sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}$, where

$$b_{j,k}(z) = \int_{z_{j,k}}^{z} \widehat{b}_{j,k}(t)dt. \tag{5.15}$$

From [122, Th 13.5 Vol I] we deduce that for each j fixed, $1 \le j \le M$, $\{b_{j,k}\}_{k=0}^{\infty}$ is a sequence of single valued analytic functions on U such that $b_{j,k}(z_{j,k}) = 0$. □

Before we state the next Theorem, we remind some concepts from analytic function continuation (cf. [122, Vol II, Ch. 8]),

DEFINITION 5.1. *A pair $\{G, f\}$ consisting of a domain G and an analytic function f on G is said an (function) element and G is the domain of the element.*

DEFINITION 5.2. *Each two elements $\{G, f\}, \{D, g\}$ is said to be a direct analytic continuation of each other if $G \bigcap D \neq \emptyset$ and if there exists a domain $\mathfrak{g} \subset G \bigcap D$ such that $f(z) = g(z), \forall z \in \mathfrak{g}$.*

From the preceding Lemma, we obtain the existence of M linearly independent holomorphic solutions for the differential equation (5.2) defined on the exterior of any closed Jordan curve containing the set $supp(\nu)$.

Proof. (of Theorem 5.1)

Let C_1 be a circle such that $int(\gamma) \subsetneq int(C_1)$. Consider the sequence of complex numbers

$$\{z_{j,k}\}_{\substack{j=1,\dots,M\\ k=0,\dots,\infty}} \subset \left[G_{\gamma,\mathfrak{r}_\zeta} \setminus ext(C_1)\right] \setminus C_1.$$

Consider a sequence of circles $\{C_n\}_{n=2}^{\infty}$ satisfying

- $int(C_1) \subsetneq int(C_n) \subsetneq int(C_{n+1})$
- $\bigcup_{n>1} int(C_n) = \mathbb{C}$.

Define the sequence of simply connected domains

$$S_n = \left[G_{\gamma,\mathfrak{r}_\zeta} \setminus ext(C_n)\right] \setminus C_n, \quad n \ge 1.$$

Notice that $S_1 \subset \cdots \subset S_n \subset \cdots$. Let us take two consecutive sets S_n, S_{n+1}. From Lemma 5.1 we have that there exist two analytic functions $y_{j,n}(z,\varepsilon) = \sum_{k=0}^{\infty} b_{j,k,n}(z)\varepsilon^k, y_{j,n+1}(z,\varepsilon) = \sum_{k=0}^{\infty} b_{j,k,n+1}(z)\varepsilon^k$ on S_n, S_{n+1}, respectively, uniquely determined from the conditions $b_{j,k,n}(z_{j,k}) = 0, b_{j,k,n+1}(z_{j,k}) = 0$. As $S_n \subset S_{n+1}$ we have that for each j,k fixed $\{S_{n+1}, b_{j,k,n+1}\}$ is a direct analytic continuation of $\{S_n, b_{j,k,n}\}$.

Consider now the set $\mathfrak{S}_{j,k} = \bigcup_{n\in\mathbb{N}} \{S_n, b_{j,k,n}\}$ with the relation of order $\{S_n, b_{j,k,n}\} < \{S_m, b_{j,k,S_m}\}$ if an only if $S_m \subset S_n$. By construction, $\mathfrak{S}_{j,k}$ is a totally ordered set and by the Zorn's Lemma there exists a maximal element $\{X, Y\}$ in $\mathfrak{S}_{j,k}$. This element is unique. To prove this assume that there exist two maximal elements $\{X_1, Y_1\}$, $\{X_2, Y_2\}$. Since $\{X_1, Y_1\} \in \mathfrak{S}_{j,k}$, $\{X_2, Y_2\}$ is maximal and $\mathfrak{S}_{j,k}$ is totally ordered we have either $\{X_1, Y_1\} < \{X_2, Y_2\}$ or $\{X_2, Y_2\} < \{X_1, Y_1\}$, which is a contradiction. Hence the maximal element is unique. It is not difficult to se that $X = G_{\gamma,\mathfrak{r}_\zeta}$ (since X must contain all the S_n) and Y is a holomorphic function on $G_{\gamma,\mathfrak{r}_\zeta}$. $\square$

The following Lemma is a direct consequence of the Cauchy formula for the multiplication of two formal power series (the discrete convolution of the two sequences) [73, (0.316)].

LEMMA 5.2. *Let* $\left\{\sum_{k=0}^{\infty} a_{\nu,k} z^{k-n_\nu}\right\}_{\nu=1}^{m}$ *be m formal power series, $n_1,\dots,n_m \in \mathbb{Z}_+$ and $n = \sum_{i=1}^{m} n_i$. Then*

$$\prod_{\nu=1}^{m}\left(\sum_{k=0}^{\infty} a_{\nu,k} z^{k-n_\nu}\right) = \sum_{k=0}^{\infty}\left(\sum_{j_1+\cdots j_m=k} a_{1,j_1}\cdots a_{m,j_m}\right) z^{k-n}.$$

Let us define the multi–indices of non negative integers,

$$\overline{n} = \{n_1,\dots,n_i\}, \overline{s} = \{s_{1,n_1},\dots,s_{t_1,n_1},\dots,s_{1,n_i},\dots,s_{t_i,n_i}\}, \overline{t} = \{t_1,\dots,t_i\}.$$

Let T be the diophantine system of equations

$$\begin{gathered} n_1 + 2n_2 \cdots + i n_i = i, \\ \max\{1, -k+i\} \le n_1 + \cdots + n_i \le i, \\ t_1 + \cdots + t_i = k + n_1 + \cdots + n_i - i, \\ s_{1,n_1} + \cdots + t_1 s_{t_1,n_1} = t_1, \\ \vdots \\ s_{1,n_i} + \cdots + t_i s_{t_i,n_i} = t_i. \end{gathered}$$

The next Lemma give us an explicit formula to calculate the functions that define the exponential series and will be used in the sequel.

LEMMA 5.3. *Let γ be a closed Jordan curve such that supp$(\nu) \subset$ int(γ), $U = G_{\gamma,\mathfrak{r}_\zeta}$, where $G_{\gamma,\mathfrak{r}_\zeta}$ is as in Theorem 5.1. If* $e^{\sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}}$ *is a solution of* (5.2) *defined on $U \times V_0^*$, then the functions $b_{j,k}^{(1)}$ satisfy the recurrence relations*

$$\begin{aligned} \left[b_{j,0}^{(1)}(z)\right]^{M} &= \frac{1}{\rho_M(z)} \\ M\left[b_{j,0}^{(1)}(z)\right]^{M-1} b_{j,k}^{(1)}(z) + R_k\left(b_{j,0}^{(1)}(z),\dots,b_{j,k-1}^{(1)}(z),\dots,b_{j,0}^{(M)}(z),\dots,b_{j,k-1}^{(M)}(z)\right) &= 0, \end{aligned} \tag{5.16}$$

$k \in \mathbb{N}$, *where* $R_k(.) = \sum_{j=0}^{\min\{M,k\}} \frac{\rho_{k-j}(z)}{\rho_M(z)} \Upsilon_{k-j,M-j}(z) - \left[b_{j,0}^{(1)}(z)\right]^{M-1} b_{j,k}^{(1)}(z)$ *doesn't depend on* $b_{j,k}^{(1)}$ *and*

$$\begin{aligned} \Upsilon_{k,i}(z) &= \sum_{\overline{n},\overline{s},\overline{t}\in T} \binom{i}{n_1,\dots,n_i} \prod_{r=1}^{i} \frac{n_r!}{(n_r-\mu(s,r))! s_{1,n_r}!\cdots s_{t_r,n_r}!} \left[b_{j,0}^{(r)}(z)\right]^{n_r-\mu(s,r)} \prod_{v=1}^{t_i} \frac{\left[b_{j,v}^{(r)}(z)\right]^{s_{v,n_r}}}{(r!)^{n_r}} \\ \mu(s,r) &= s_{1,n_r} + \cdots + s_{k,n_r}, \end{aligned}$$

and

$$\begin{aligned}\left[b_{j,0}^{(1)}(z)\right]^M &= \frac{1}{\rho_M(z)}, \\ b_{j,1}^{(1)}(z) &= \frac{M-1}{2M}\frac{\rho_M^{(1)}(z)}{\rho_M(z)} - \frac{1}{M}\frac{\rho_{M-1}(z)}{\rho_M(z)}.\end{aligned} \tag{5.17}$$

Proof. From Lemma 5.1 there exist M solutions to the equation (5.2) of the form

$$u_j(z,\varepsilon) = e^{y_j(z,\varepsilon)} \quad \text{where} \quad y_j(z,\varepsilon) = \sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}.$$

Relations (5.7),(5.8) and the transformation of the variable $u = e^y$ give us that (5.2) can be expressed as

$$\sum_{i=1}^{M} \varepsilon^M P_i(y^{(1)}, \dots, y^{(i)}) \frac{\rho_i(z)}{\rho_M(z)} - \frac{1}{\rho_M(z)} = 0. \tag{5.18}$$

From (5.7), [171, page 237] we have,

$$\varepsilon^M P_i\left(\left(\sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}\right)^{(1)}, \dots, \left(\sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon^{k-1}\right)^{(i)}\right) =$$

$$\varepsilon^M \sum_{\substack{n_1+\cdots+in_i=i \\ 1\le n_1+\cdots+n_i\le i}} \binom{i}{n_1,\dots,n_i} \prod_{r=1}^{i} \left(\frac{\sum_{k=0}^{\infty} \left(m_{k,n_r}(b_{j,0}^{(r)}(z),\dots,b_{j,k}^{(r)}(z))\right)\varepsilon^{k-n_r}}{(r!)^{n_r}} \right), \tag{5.19}$$

where

$$m_{k,n_r}(b_{j,0}^{(r)}(z),\dots,b_{j,k}^{(r)}(z)) =$$
$$\sum_{s_{1,n_r}+\cdots+ks_{k,n_r}=k} \frac{n_r!}{(n_r-\mu(s,r))!s_{1,n_r}!\cdots s_{k,n_r}!} \left[b_{j,0}^{(r)}(z)\right]^{n_r-\mu(s,r)} \prod_{v=1}^{k} \left[b_{j,v}^{(r)}(z)\right]^{s_{v,n_r}},$$

and $\mu(s,r) = s_{1,n_r} + \cdots + s_{k,n_r}$.

From Lemma 5.2 we deduce that (5.19) is

$$\varepsilon^M \sum_{\substack{n_1+\cdots+in_i=i \\ 1\le n_1+\cdots+n_i\le i}} \binom{i}{n_1,\dots n_i} \sum_{k=0}^{\infty} \sum_{t_1+\cdots+t_i=k} \prod_{r=1}^{i} \frac{m_{t_r,n_r}(b_{j,0}^{(r)}(z),\dots,b_{j,t_r,U}^{(r)}(z))}{(r!)^{n_r}} \varepsilon^{k-n_1-\cdots-n_i} =$$

$$\varepsilon^M \left(\sum_{\overline{n},\overline{t},k\in T_1} \binom{i}{n_1,\dots n_i} \sum_{\overline{s},\overline{t}\in T_2} \prod_{r=1}^{i} \frac{n_r!}{(n_r-\mu(s,r))!s_{1,n_r}!\cdots s_{t_r,n_r}!} \frac{\left[b_{j,0}^{(r)}(z)\right]^{n_r-\mu(s,r)}}{(r!)^{n_r}} \prod_{v=1}^{t_r} \left[b_{j,v}^{(r)}(z)\right]^{s_{v,n_r}} \right) \varepsilon^k,$$

where T_1 is the system

$$\begin{aligned} n_1 + \cdots + i n_i &= i, \\ 1 \leq n_1 + \cdots + n_i &\leq i, \\ -n_1 - \cdots - n_i \leq k &\leq \infty, \\ t_1 + \cdots + t_i &= k + n_1 + \cdots + n_i, \end{aligned}$$

and T_2 is the system

$$\begin{aligned} s_{1,n_1} + \cdots + t_1 s_{t_1,n_1} &= t_1, \\ &\vdots \\ s_{1,n_i} + \cdots + t_i s_{t_i,n_i} &= t_i. \end{aligned}$$

Therefore, (5.19) simplifies to

$$\varepsilon^M \sum_{k=-i}^{\infty} \left(\sum_{\overline{n},\overline{t},k \in T_3} \binom{i}{n_1,\dots n_i} \sum_{\overline{s},\overline{t} \in T_2} \prod_{r=1}^{i} \frac{n_r!}{(n_r-\mu(s,r))! s_{1,n_r}! \cdots s_{t_r,n_r}!} \frac{\left[b_{j,0}^{(r)}(z)\right]^{n_r-\mu(s,r)}}{(r!)^{n_r}} \prod_{v=1}^{t_r} \left[b_{j,v}^{(r)}(z)\right]^{s_{v,n_r}} \right) \varepsilon^k,$$

where T_3 is the system

$$\begin{aligned} n_1 + \cdots + i n_i = i, \\ \max\{1,-k\} \leq n_1 + \cdots + n_i \leq i, \\ t_1 + \cdots + t_i = k + n_1 + \cdots + n_i. \end{aligned}$$

From this we deduce that

$$\varepsilon^M P_i \left(\left(\sum_{k=0}^{\infty} b_{j,k}(z) \varepsilon^{k-1} \right)^{(1)}, \dots, \left(\sum_{k=0}^{\infty} b_{j,k}(z) \varepsilon^{k-1} \right)^{(i)} \right) = \varepsilon^M \sum_{k=0}^{\infty} \Upsilon_{k,i} \varepsilon^{k-i},$$

where

$$\Upsilon_{k,i} = \sum_{\overline{n},\overline{s},\overline{t} \in T} \binom{i}{n_1,\dots,n_i} \prod_{r=1}^{i} \frac{n_r!}{(n_r-\mu(s,r))! s_{1,n_r}! \cdots s_{t_r,n_r}!} \left[b_{j,0}^{(r)}(z)\right]^{n_r-\mu(s,r)} \prod_{v=1}^{t_r} \frac{\left[b_{j,v}^{(r)}(z)\right]^{s_{v,n_r}}}{(r!)^{n_r}}. \tag{5.20}$$

We have then that (5.18) can be expressed as

$$-\frac{1}{\rho_M(z)} + \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\min\{M-1,k\}} \frac{\rho_{M-j}(z)}{\rho_M(z)} \Upsilon_{k-j,M-j} \right) \varepsilon^k = 0. \tag{5.21}$$

From the equality to 0 of the power series in ε (5.21), we deduce that each coefficient of the power series must equal 0.

Consider now the coefficient associated to the power ε^k of (5.21). We want to show that

$$\Upsilon_{k,M}(z) + \frac{\rho_{M-1}(z)}{\rho_M(z)} \Upsilon_{k-1,M-1}(z) + \cdots + \frac{\rho_{M-j}(z)}{\rho_M(z)} \Upsilon_{k-j,M-j}(z),$$

is of the form

$$M\left[b^{(1)}_{j,0}(z)\right]^{M-1} b^{(1)}_{j,k}(z) + R_k\left(b^{(1)}_{j,0}(z),\ldots,b^{(1)}_{j,k-1}(z),\ldots,b^{(M)}_{j,0}(z),\ldots,b^{(M)}_{j,k-1}(z)\right),$$

and $R_k(.)$ does not depend on $b^{(1)}_{j,k}(z)$. To prove it, we shall show that $\Upsilon_{k,M}$ contains the summand $M\left[b^{(1)}_{j,0}(z)\right]^{M-1} b^{(1)}_{j,k}(z)$ and this summand will not be present in the remaining terms $\Upsilon_{k-j,M-j}$. We analyze separately the cases $j=0$ and $0<j\le\min\{M-1,k\}$.

Suppose that $j=0$. From the set of diophantine equations that define the sum in (5.20) we have that if $i=M$ and $n_1=M$ then $t_1=k$, taking the partition $s_{t_1,M}=1$ corresponding to $t_1=k$ we have that $\rho_{k,M}$ contains the summand $M\left[b^{(1)}_{j,0}\right]^{M-1} b^{(1)}_{j,k}$. It is clear that if we take another partition $\{s_{1,n_1},\ldots,s_{t_1,n_1}\}$ corresponding to $t_1=k$ it will not contain the term $b^{(1)}_{j,k}$.

It is also clear that the only solution to the diophantine equation $n_1+\cdots+Mn_M=M$ for which $n_1+\cdots+n_M=M$ is $n_1=M$ which implies that $t_1+\cdots t_M<k$. Therefore, the remaining summands of $\rho_{k,M}$ do not contain the term $b_{j,k}$ nor its higher derivatives.

We analyze now the case $0<j\le\min\{M-1,k\}$. Again, from the set of diophantine equations that define the sum in (5.20) if $j>0$ we have that $t_1+\cdots+t_{M-j}=k-j+n_1+\cdots+n_{M-j}-(M-j)<k$, therefore the remaining summands of $\Upsilon_{k-j,M-j}$ do not contain the term $b^{(1)}_{j,k}$ nor its higher derivatives.

The particular cases are of interest, the coefficients of (5.21) associated to the power $k=0$ and $k=1$ of ε^k. We have then

$$\begin{aligned}\Upsilon_{0,M}(z) &= \frac{1}{\rho_M(z)},\\ \frac{\rho_{M-1}(z)}{\rho_M(z)}\Upsilon_{0,M-1}(z)+\Upsilon_{1,M}(z) &= 0.\end{aligned}$$

Expression (5.20) gives

$$\begin{aligned}\Upsilon_{1,M}(z) &= \frac{M(M-1)}{2}\left[b^{(1)}_{j,0}(z)\right]^{M-2} b^{(2)}_{j,0}(z) + M\left[b^{(1)}_{j,0}(z)\right]^{M-1} b^{(1)}_{j,1}(z),\\ \Upsilon_{0,M-1}(z) &= \left[b^{(1)}_{j,0}(z)\right]^{M-1},\end{aligned}$$

which lead us to the equations (5.17). □

5.3 Strong asymptotic behavior

The results of the preceding section can be applied to find the strong asymptotic behavior of the eigenpolynomials of the operator under the assumption that ρ_M is a real polynomial. Let γ be a closed Jordan curve such that $supp(\nu)\subset int(\gamma)$, from the assumption that ρ_M is real, it is immediate that $\min_{z\in\gamma\bigcap\mathbb{R}} z$ is finite. In the sequel, let us define $\zeta=\min_{z\in\gamma\bigcap\mathbb{R}} z$, and $\mathfrak{r}_\zeta=(-\infty,\zeta)$.

From (5.15) and Theorem 5.1 we deduce that

$$b_{j,0}(z)=\int_{z_0}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}}+c,\quad j=1,\ldots,M,$$

where $z,z_0\in G_{\gamma,\mathfrak{r}_\zeta}$, $c\in\mathbb{C}$.

For a given region $G_{\gamma,\mathfrak{r}_\zeta}$, we shall denote by $b_{1,0}$ defined for $z \in G_{\gamma,\mathfrak{r}_\zeta}$ a primitive, with z_0 and c real numbers, of the branch of the function $\dfrac{1}{\sqrt[M]{\rho_M(z)}}$ which coincides with $\dfrac{1}{z}$ at ∞. The primitive on the remaining sheets will be denoted by $b_{j,0}$. In this section we obtain a formula for the strong asymptotic behavior of the polynomial Q_n. We need some preliminaries lemmas.

LEMMA 5.4. *Assume that ρ_M is a real monic polynomial and define the function $g_1(x) = b_{1,0}(x) - \ln x, x > 0$. There exists a $x_0 \in \mathbb{R}^+$ such that if $x \geq x_0$ then*

$$2\left|g_1(x-\delta) - g_1(x)\right| < \left(\operatorname{Re}\left[e^{\imath\dfrac{2(j-1)\pi}{M}}\right] - 1\right)(\ln(x-\delta) - \ln x), \quad j = 2, \ldots, M, \tag{5.22}$$

for $0 < \delta < \delta_0(x)$, for some $\delta_0(x)$.

Proof. As ρ_M is a real polynomial we have two alternatives; either there exists a point $x_1 \in \mathbb{R}^+$ large enough such that

$$g_1^{'}(x) = \frac{1}{\sqrt[M]{\rho_M(x)}} - \frac{1}{x} > 0, \quad \forall x \geq x_1, \quad j = 2, \ldots, M, \tag{5.23}$$

or

$$g_1^{'}(x) = \frac{1}{\sqrt[M]{\rho_M(x)}} - \frac{1}{x} < 0, \quad \forall x \geq x_1, \quad j = 2, \ldots, M. \tag{5.24}$$

Suppose that (5.23) holds, then it is easy to see that there exists a x_0 such that if $x \geq x_0$ then

$$\frac{1 + \operatorname{Re}\left[e^{\imath\dfrac{2(j-1)\pi}{M}}\right]}{2}\sqrt[M]{\rho_M(x)} < \sqrt[M]{\rho_M(x)} < x,$$

which is equivalent to

$$2\left(\frac{1}{x} - \frac{1}{\sqrt[M]{\rho_M(x)}}\right) < \frac{1 - \operatorname{Re}\left[e^{\imath\dfrac{2(j-1)\pi}{M}}\right]}{x}, \quad j = 2, \ldots, M. \tag{5.25}$$

Suppose that (5.24) holds, we have that there exists a x_0 such that if $x \geq x_0$ then

$$x < \sqrt[M]{\rho_M(x)} < \frac{3 - \operatorname{Re}\left[e^{\imath\dfrac{2(j-1)\pi}{M}}\right]}{2}\sqrt[M]{\rho_M(x)},$$

and this gives

$$2\left(\frac{1}{\sqrt[M]{\rho_M(x)}} - \frac{1}{x}\right) < \frac{1 - \operatorname{Re}\left[e^{\imath\dfrac{2(j-1)\pi}{M}}\right]}{x}, \quad j = 2, \ldots, M. \tag{5.26}$$

From (5.25),(5.26) we deduce that there exists $x_0 \in \mathbb{R}^+$ such that

$$2|g_1'(x)| < \frac{1 - \mathrm{Re}\left[e^{\imath\frac{2(j-1)\pi}{M}}\right]}{x}, \quad j = 2, \ldots, M, \tag{5.27}$$

for $x > x_0$. From the relations

$$\begin{aligned} |g_1'(x)| &= \lim_{\delta\to 0} \frac{|g_1(x-\delta) - g_1(x)|}{\delta}, \\ -\frac{1}{x} &= \lim_{\delta\to 0} \frac{\ln(x-\delta) - \ln(x)}{\delta}, \end{aligned}$$

and from (5.27) we deduce (5.22). □

LEMMA 5.5. *Assume that ρ_M is a real monic polynomial. There exists a $x_0 \in \mathbb{R}^+$ such that if $x \geq x_0$ then*

$$b_{1,0}(x-\delta) - b_{1,0}(x) < \mathrm{Re}[b_{j,0}(x-\delta)] - \mathrm{Re}[b_{j,0}(x)], \quad \forall j > 1, \tag{5.28}$$

for $0 < \delta < \delta_0(x)$, for some $\delta_0(x)$.

Proof. Let x_a be such that $\rho_M(x) > 0, \forall x > x_a$. Then $\mathrm{Re}[b_{j,0}(x)] = \mathrm{Re}\left[e^{\imath\frac{2(j-1)\pi}{M}}\right] b_{1,0}(x)$ for $j = 2, \ldots, M$ and $x > x_a$.

As $\mathrm{Re}\left[e^{\imath\frac{2(j-1)\pi}{M}}\right] < 1, \forall j > 1$, then the following inequality holds,

$$\begin{aligned} & -|b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)| \\ < \;& \mathrm{Re}\left[b_{j,0}(x-\delta) - b_{j,0}(x)\right] - \mathrm{Re}\left[e^{\imath\frac{2(j-1)\pi}{M}}\{\ln(x-\delta) - \ln x\}\right], \end{aligned}$$

for $x > x_a, 0 < \delta < \delta_a(x)$, from where we deduce,

$$\begin{aligned} \mathrm{Re}\left[e^{\imath\frac{2(j-1)\pi}{M}}\{\ln(x-\delta) - \ln x\}\right] - |b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)| \quad < \\ \mathrm{Re}\left[b_{j,0}(x-\delta) - b_{j,0}(x)\right], \quad j = 2, \ldots, M, \end{aligned} \tag{5.29}$$

is valid for $x > x_a$.

Let us denote $h(x) = \mathrm{Re}\left[e^{\imath\frac{2(j-1)\pi}{M}} - 1\right]\{\ln(x-\delta) - \ln x)\}$ for $j = 2, \ldots, M$. From (5.22) of Lemma 5.4 we deduce that there exists $x_1 \in \mathbb{R}^+$ such that

$$h(x) > 2\left|b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)\right|, \tag{5.30}$$

for $x > x_1, 0 < \delta < \delta_1(x)$.

Let x_0 be such that, $x_0 > \max\{x_a, x_1\}$ and $\delta(x) < \min\{\delta_a(x), \delta_1(x)\}$ such that (5.29) and (5.30) hold simultaneously for $x > x_0, 0 < \delta < \delta(x)$.

From

$$b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x) \leq |b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)|,$$

we deduce that

$$b_{1,0}(x-\delta) - b_{1,0}(x) - 2\,|b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)| \leq$$

$$\ln(x-\delta) - \ln x - |b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)|.$$

This implies

$$b_{1,0}(x-\delta) - b_{1,0}(x) - 2\,|b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)| + h(x) \leq$$

$$\ln(x-\delta) - \ln x + h(x) - |b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)| =$$

$$\mathrm{Re}[e^{\imath\frac{2(j-1)\pi}{M}}\{\ln(x-\delta) - \ln x\}] - |b_{1,0}(x-\delta) - b_{1,0}(x) - (\ln(x-\delta) - \ln x)| <$$

$$\mathrm{Re}[b_{j,0}(x-\delta) - b_{j,0}(x)],$$

where the last inequality follows from (5.29). From first inequality and the last inequality of the above expression and from (5.30) we deduce that

$$b_{1,0}(x-\delta) - b_{1,0}(x) < \mathrm{Re}[b_{j,0}(x-\delta)] - \mathrm{Re}[b_{j,0}(x)], \quad \forall j > 1,$$

for $x \geq x_0, 0 < \delta < \delta_0(x)$, for some $\delta_0(x)$, which is (5.28). □

LEMMA 5.6. *Assume that $a_0, \dots, a_M \in \mathbb{C}, a_0 \neq 0$ and consider the algebraic equation*

$$\sum_{k=1}^{M} a_k \omega^k - \frac{a_0}{\varepsilon^M} = 0. \tag{5.31}$$

Then for every j fixed, $j = 0, \dots, M-1$, there exists a unique analytic function

$$\omega_j(\varepsilon) = \sum_{k=0}^{\infty} h_{j,k}\varepsilon^{k-1}, \quad h_{j,0} = \sqrt[M]{|a_0|}e^{\imath\Im\left[\frac{\arg[a_0] + 2\,j\,\pi}{M}\right]},$$

defined on some reduced neighborhood $V_\varepsilon^*(0)$ of $\varepsilon = 0$ such that $\omega_j(\varepsilon)$ is a root of (5.31).

Proof. Without loss of generality we shall assume that $a_M = 1$. Multiplying by ε and making the change of variable $w = \omega\varepsilon$, (5.31) becomes into

$$F(\varepsilon, w) = w^M + \sum_{k=1}^{M-1} a_k \varepsilon^{M-k} w^k - a_0 = 0. \tag{5.32}$$

Notice that $F(0, w) = w^M - a_0$ has M roots. Denote by $w_{j,0}$ the roots of $\sqrt[M]{a_0}$, we have then

$$w_{j,0} = \sqrt[M]{|a_0|}e^{\imath\Im\left[\dfrac{\arg[a_0]+2\,j\,\pi}{M}\right]}, \quad j = 0,\ldots,M-1.$$

Let us fix j, and notice that $\dfrac{\partial F}{\partial w}(0, w_{j,0}) \neq 0$. From the implicit function theorem [122, Th 3.11, Vol II] there exists an unique analytic function $w_j(\varepsilon) = \displaystyle\sum_{k=0}^{\infty} \varepsilon^k h_{j,k}$ such that $w_j(0) = h_{j,0} = w_{j,0}$, on some neighborhood $V_\varepsilon(0)$ of $\varepsilon = 0$ which is a solution to (5.32).

Taking into account that $w = \omega\varepsilon, \varepsilon \neq 0$ we obtain that $\omega_j(\varepsilon) = \displaystyle\sum_{k=0}^{\infty} \varepsilon^{k-1} h_{j,k}$ is a solution of (5.31) for some reduced neighborhood $V_\varepsilon^*(0)$ of $\varepsilon = 0$. □

From the preceding lemmas we obtain the following representation for the polynomial Q_n on the exterior of any closed Jordan curve γ containing $supp(\nu)$.

LEMMA 5.7. *Assume that ρ_M is a real monic polynomial and γ a closed Jordan curve such that $supp(\nu) \subset int(\gamma)$. Define $y_1(z, \varepsilon_n)$ on $G_{\gamma,\mathfrak{r}_\zeta}$ as in Theorem 5.1. Then, there exists $n_0 \in \mathbb{N}$ and an unique constant c_n such that*

$$Q_n(z) = c_n e^{y_1(z,\varepsilon_n)}, \quad \forall n > n_0, \quad \forall z \in G_{\gamma,\mathfrak{r}_\zeta}, \tag{5.33}$$

where $\varepsilon_n^M = \dfrac{1}{\lambda_n - \rho_{0,0}}$ and Q_n a monic polynomial eigenfunction of (5.1).

Proof. According to [15], there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$ the operator (5.1) has a unique polynomial eigenfunction Q_n for every fixed n with eigenvalue $\lambda_n = \displaystyle\sum_{0\leq k\leq \min(M,n)} \rho_{k,k}\frac{n!}{(n-k)!}$ and with zeros contained on $int(\gamma)$. We have then that Q_n is a polynomial solution to (5.2) with $\varepsilon_n^M = \dfrac{1}{\lambda_n - \rho_{0,0}}$.

From Theorem 5.1 we deduce that there exist M sequences $\{b_{j,k}\}_{k=0}^{\infty}$ of single–valued analytic functions satisfying (5.16) and unique constants $c_1(n), \ldots, c_M(n)$ such that

$$Q_n(z) = c_1(n)e^{y_1(z,\varepsilon_n)} + \cdots + c_M(n)e^{y_M(z,\varepsilon_n)}, \quad \forall z \in G_{\gamma,\mathfrak{r}_\zeta},$$

where $y_j(z, \varepsilon_n) = \displaystyle\sum_{k=0}^{\infty} b_{j,k}(z)\varepsilon_n^{k-1}$.

By Lemma 5.5 there exists a $x_0 \in \mathbb{R}^+$ large enough (in particular it can be assumed that $x_0 \in G_{\gamma,\mathfrak{r}_\zeta}$) and $\delta(x_0) > 0$ such that

$$b_{1,0}(x_0 - \delta) - b_{1,0}(x_0) < \mathrm{Re}[b_{j,0}(x_0 - \delta)] - \mathrm{Re}[b_{j,0}(x_0)], \quad \forall j > 1,$$

for every δ such that $0 < \delta < \delta_0(x)$.

From [15, Lemma 9] we have that for n sufficiently large the zeros of Q_n are contained in a compact set K. Consider an open subset $U \subset K \subset G_{\gamma,\mathfrak{r}_\zeta}$ such that $\{x_0, x_0 - \delta\} \in U$. From [15, Ths. 2,4] and the relation $\dfrac{1}{n}\dfrac{Q_n'(z)}{Q_n(z)} = \displaystyle\int \frac{d\nu_n(\zeta)}{z-\zeta}$, we deduce that,

$$\lim_{n\to\infty} \sqrt[n]{|Q_n(z)|} = e^{\mathrm{Re}[b_{1,0}(z) + c]}, \quad \forall z \in U \tag{5.34}$$

for some $c \in \mathbb{R}$. Then, we have

$$\begin{aligned} Q_n(x_0) &= c_1(n)e^{y_1(x_0,\varepsilon_n)} + \quad \dots \quad + c_M(n)e^{y_M(x_0,\varepsilon_n)} \\ &\vdots \\ Q_n^{(M-1)}(x_0) &= c_1(n)\frac{D^{M-1}}{dx^M}\left(e^{y_1(x,\varepsilon_n)}\right)\Big|_{x=x_0} + \quad \dots \quad + c_M(n)\frac{D^{M-1}}{dx^{M-1}}\left(e^{y_M(x,\varepsilon_n)}\right)\Big|_{x=x_0}. \end{aligned}$$

Using Cramer's rule we obtain the following representation for the coefficients $c_j(n)$,

$$c_j(n) = \frac{W_{x_0}(\dots, e^{y_{j-1}(x,\varepsilon_n)}, Q_n(x), e^{y_{j+1}(x,\varepsilon_n)}, \dots)}{W_{x_0}(e^{y_1(x,\varepsilon_n)}, \dots, e^{y_M(x,\varepsilon)})}, \tag{5.35}$$

where $W_{x_0}(f_1(x), \dots, f_n(x))$ denotes the Wronskian of the set of functions $\{f_1, \dots, f_n\}$ at $x = x_0$.

Denote by j_0 the index satisfying,

$$\begin{aligned} \mathrm{Re}[(b_{j_0,0}(x_0-\delta) - b_{j_0,0}(x_0))] &= \max_{j=1,\dots,M}\{\mathrm{Re}[b_{j,0}(x_0-\delta) - b_{j,0}(x_0))]\} \\ c_{j_0}(n) &\neq 0. \end{aligned} \tag{5.36}$$

According to Lemma 5.5, $j_0 \neq 1$.

Let us express $Q_n(x)$ at $x = x_0 - \delta$ as

$$\begin{aligned} &\mu_{j_0}(x_0,\varepsilon_n)d_{j_0}(x_0-\delta,\varepsilon_n) \\ &\qquad e^{\mathrm{Re}[y_{j_0}(x_0-\delta,\varepsilon_n) - y_{j_0}(x_0,\varepsilon_n)]}\left(1 + \sum_{j_k \neq j_0} \frac{\mu_{j_k}(x_0,\varepsilon_n)d_{j_k}(x_0-\delta,\varepsilon_n)}{\mu_{j_0}(x_0,\varepsilon_n)d_{j_0}(x_0-\delta,\varepsilon_n)} e^{\Theta_{j_k}(x_0,x_0-\delta)}\right), \end{aligned} \tag{5.37}$$

where

$$\begin{aligned} \Theta_{j_k}(x_0, x_0-\delta) &= \mathrm{Re}\left[y_{j_k}(x_0-\delta,\varepsilon_n) - y_{j_k}(x_0,\varepsilon_n) - (y_{j_0}(x_0-\delta,\varepsilon_n) - y_{j_0}(x_0,\varepsilon_n))\right], \\ \mu_j(x_0,\varepsilon_n) &= c_j(n)e^{y_j(x_0,\varepsilon_n)}, \\ d_j(x_0-\delta,\varepsilon_n) &= e^{\Im[y_j(x_0-\delta,\varepsilon_n) - y_j(x_0,\varepsilon_n)]}. \end{aligned}$$

We claim that $\lim_{n\to\infty}\left|\frac{\mu_{j_k}(x_0,\varepsilon_n)}{\mu_{j_0}(x_0,\varepsilon_n)}\right|$ exists and is finite. Indeed, from the definition of $\mu_j(x_0,\varepsilon_n)$ and (5.35) the term $\frac{\mu_{j_k}(x_0,\varepsilon_n)}{\mu_{j_0}(x_0,\varepsilon_n)}$ can be written as

$$\frac{\alpha_{j_k,0}(x_0,\varepsilon_n)Q_n(x_0) + \dots + \alpha_{j_k,M-1}(x_0,\varepsilon_n)Q_n^{(M-1)}(x_0)}{\alpha_{j_0,0}(x_0,\varepsilon_n)Q_n(x_0) + \dots + \alpha_{j_0,M-1}(x_0,\varepsilon_n)Q_n^{(M-1)}(x_0)}, \tag{5.38}$$

where $\alpha_{j_k,k}(x_0,\varepsilon_n)$ is the minor associated to the element $Q_n^{(k)}(x_0)$ of the matrix

$$\begin{pmatrix} \dots & 1 & Q_n(x_0) & 1 & \dots \\ \dots & \frac{\frac{D}{dx}\left(e^{y_{j_k-1}(x,\varepsilon_n)}\right)\big|_{x=x_0}}{e^{y_{j_k-1}(x_0,\varepsilon_n)}} & Q_n'(x_0) & \frac{\frac{D}{dx}\left(e^{(y_{j_k+1}(x,\varepsilon_n))}\right)\big|_{x=x_0}}{e^{y_{j_k+1}(x_0,\varepsilon_n)}} & \dots \\ & & \vdots & & \\ \dots & \frac{\frac{D^{M-1}}{dx^{M-1}}\left(e^{y_{j_k-1}(x,\varepsilon_n)}\right)\big|_{x=x_0}}{e^{y_{j_k-1}(x_0,\varepsilon_n)}} & Q_n^{(M-1)}(x_0) & \frac{\frac{D^{M-1}}{dx^{M-1}}\left(e^{y_{j_k-1}(x,\varepsilon_n)}\right)\big|_{x=x_0}}{e^{y_{j_k-1}(x_0,\varepsilon_n)}} & \dots \end{pmatrix}.$$

Notice that

$$\alpha_{j_k,M-1}(x_0,\varepsilon_n) = W_{x_0}\left(\dots, e^{y_{j_k-1}(x,\varepsilon_n)}, e^{y_{j_k+1}(x,\varepsilon_n)}, \dots\right) e^{-\sum\limits_{j\neq j_k} y_j(x_0,\varepsilon_n)}, \tag{5.39}$$

from where we deduce that

$$\alpha_{j_k,M-1}(x_0,\varepsilon_n) \neq 0. \tag{5.40}$$

According to [15] we have,

$$\lim_{n\to\infty} \frac{Q_n^{(k)}(x_0)}{n\cdots(n-k+1)Q_n(x_0)} = \frac{1}{\rho_M(x_0)}. \tag{5.41}$$

From Theorem 5.1 $y_j(z,\varepsilon_n) = \sum\limits_{k=0}^{\infty} b_{j,k}(z)\varepsilon_n^{k-1}, b_{j,k} \in \mathcal{H}(U)$, hence from (5.39) and (5.7) we have

$$\lim_{n\to\infty} \frac{\alpha_{j_k,M-1}(x_0,\varepsilon_n)}{n^{M^2(M-1)/2}} = (-1)^{M-1+j_k} \begin{vmatrix} \cdots & 1 & 1 & \cdots \\ \cdots & b'_{j_k-1,0}(x_0) & b'_{j_k+1,0}(x_0) & \cdots \\ & \vdots & \vdots & \\ \cdots & \left[b'_{j_k-1,0}(x_0)\right]^{M-1} & \left[b'_{j_k+1,0}(x_0)\right]^{M-1} & \cdots \end{vmatrix}. \tag{5.42}$$

From (5.40), (5.41) and (5.42), we deduce

$$\begin{aligned} \lim_{n\to\infty} \frac{\mu_{j_k}(x_0,\varepsilon_n)}{\mu_{j_0}(x_0,\varepsilon_n)} &= \frac{\dfrac{\alpha_{j_k,1}(x_0,\varepsilon_n)}{n^{M^2(M-2)/2}n^{M-1}} + \cdots + \dfrac{\alpha_{j_k,M-1}(x_0,\varepsilon_n)}{n^{M^2(M-2)/2}} \dfrac{Q_n^{(M-1)}(x_0)}{n^{M-1}Q_n(x_0)}}{\dfrac{\alpha_{j_0,1}(x_0,\varepsilon_n)}{n^{M^2(M-2)/2}n^{M-1}} + \cdots + \dfrac{\alpha_{j_0,M-1}(x_0,\varepsilon_n)}{n^{M^2(M-2)/2}} \dfrac{Q_n^{(M-1)}(x_0)}{n^{M-1}Q_n(x_0)}} \\ &= (-1)^{j_k-j_0} \frac{\begin{vmatrix} \cdots & 1 & 1 & \cdots \\ \cdots & b'_{j_k-1,0}(x_0) & b'_{j_k+1,0}(x_0) & \cdots \\ & \vdots & \vdots & \\ \cdots & \left[b'_{j_k-1,0}(x_0)\right]^{M-1} & \left[b'_{j_k+1,0}(x_0)\right]^{M-1} & \cdots \end{vmatrix}}{\begin{vmatrix} \cdots & 1 & 1 & \cdots \\ \cdots & b'_{j_0-1,0}(x_0) & b'_{j_0+1,0}(x_0) & \cdots \\ & \vdots & \vdots & \\ \cdots & \left[b'_{j_0-1,0}(x_0)\right]^{M-1} & \left[b'_{j_0+1,0}(x_0)\right]^{M-1} & \cdots \end{vmatrix}}. \end{aligned} \tag{5.43}$$

Condition (5.36) yields

$$\lim_{n\to\infty} e^{\Theta_{j_k}(x_0,x_0-\delta)} = 0, \quad j_k \neq j_0, \tag{5.44}$$

and (5.43), (5.44),

$$\lim_{n\to\infty}\left|1+\sum_{j_k\neq j_0}\frac{\mu_{j_k}(x_0,\varepsilon_n)d_{j_k}(x_0-\delta,\varepsilon_n)}{\mu_{j_0}(x_0,\varepsilon_n)d_{j_0}(x_0-\delta,\varepsilon_n)}e^{\Theta_{j_k}(x_0,x_0-\delta)}\right|^{1/n}=1. \tag{5.45}$$

It is not difficult to see that,

$$\lim_{n\to\infty}\sqrt[n]{|\mu_{j_0}(x_0,\varepsilon_n)|}=e^{\mathrm{Re}[b_{j_0,0}(x_0)]}. \tag{5.46}$$

Therefore, from (5.37), (5.43), (5.45) and (5.46) we have that

$$\lim_{n\to\infty}\sqrt[n]{|Q_n(x_0-\delta)|}=e^{\mathrm{Re}[b_{j_0,0}(x_0-\delta)]},$$

for every δ such that $0<\delta<\delta_0(x)$, which contradicts (5.34), since $j_0\neq 1$, therefore $c_j(n)=0, j>1$, from where we deduce (5.33). □

It is possible to show that the representation (5.33) is also valid on $ext(\gamma)$, as proves the following lemma

LEMMA 5.8. *With the same hypothesis of Lemma 5.7, if* $z_0\in(-\infty,\zeta)$ *then the following limit exists*

$$\lim_{\substack{z\to z_0\\ \Im(z)>0}} y_1(z,\varepsilon_n),$$

and if $y_1(z_0,\varepsilon_n)=\lim_{\substack{z\to z_0\\ \Im(z)>0}} y_1(z,\varepsilon_n)$ *then,*

$$Q_n(z)=c_n e^{y_1(z,\varepsilon_n)},\forall n>n_0 \quad \forall z\in ext(\gamma).$$

Proof. From (5.33) we deduce that for n sufficiently large, the zeros of Q_n are on $int(\gamma)$ and that,

$$e^{\ln Q_n(z)-\ln c_n+2\,k\,\pi}=e^{y_1(z,\varepsilon_n)},\forall n>n_0,\quad \forall z\in G_{\gamma,\mathfrak{r}_\zeta},\quad k\in\mathbb{Z},$$

where $\ln(z)$ is the principal value of the logarithm. From the continuity of the logarithm on the region $\Im(z)\geq 0, \mathrm{Re}(z)<\zeta$, $\lim_{\substack{z\to z_0\\ \Im(z)>0}}\ln Q_n(z)$ exists, is finite and it holds that

$$\ln Q_n(z_0)-\ln c_n+2\,k\,\pi=\lim_{\substack{z\to z_0\\ \Im(z)>0}} y_1(z,\varepsilon_n),$$

for some $k\in\mathbb{Z}$. Hence if

$$y_1(z_0,\varepsilon_n)=\lim_{\substack{z\to z_0\\ \Im(z)>0}} y_1(z,\varepsilon_n),$$

then

$$Q_n(z)=c_n e^{y_1(z,\varepsilon_n)},\forall n>n_0 \quad \forall z\in ext(\gamma).$$

□

It is not difficult to see that from the definition of $b_{1,0}$ the existence of the limit $\lim_{\substack{z\to z_0\\ \Im(z)>0}} b_{1,0}(z,\varepsilon_n)$, for $z_0\in(-\infty,\zeta)$ and from (5.16) of Lemma 5.3 we obtain the existence of $\lim_{\substack{z\to z_0\\ \Im(z)>0}} b_{1,k}(z,\varepsilon_n), k\geq 1$ for $z_0\in(-\infty,\zeta)$.

Then, according to Lemma 5.8, for $z_0\in(-\infty,\zeta)$ we define

$$b_{1,k}(z_0, \varepsilon_n) = \lim_{\substack{z \to z_0 \\ \Im(z)>0}} b_{1,k}(z, \varepsilon_n) \quad k \geq 0. \tag{5.47}$$

From the preceding lemmas we get,

Proof. (*of Theorem 5.2*)

a)

Let γ be a closed Jordan curve such that $supp(\nu) \subset int(\gamma)$. By Lemma 5.8 there exists $n_0 \in \mathbb{N}$, and a constant c_n such that,

$$Q_n(z) = c_n e^{y_1(z, \varepsilon_n)}, \forall n > n_0, \quad \forall z \in ext(\gamma),$$

where $y_1(z, \varepsilon) = \sum_{k=0}^{\infty} b_{1,k}(z)\varepsilon^{k-1}$ and $\{b_{1,k}\}_{k=0}^{\infty}$ is a sequence of single–valued functions on $ext(\gamma)$ and holomorphic on $G_{\gamma,\mathfrak{r}_\varsigma}$ satisfying (5.16).

Let's prove that there exists a sequence of functions $\{\Phi_k\}_{k=0}^{\infty}$ holomorphic on $G_{\gamma,\mathfrak{r}_\varsigma}$ such that

$$\exp\left(\sum_{k=0}^{\infty} \Phi_k(z)\varepsilon_n^{k-1}\right),$$

is the monic eigenpolynomial $Q_n(z), \forall z \in ext(\gamma)$.

Since $\exp\left(\sum_{k=0}^{\infty} b_{1,k}(z)\varepsilon_n^{k-1}\right)$ coincides with the polynomial $\frac{1}{c_n}Q_n(z)$ on $z \in ext(\gamma)$, then for n fixed we have that

$$\lim_{z\to\infty} \frac{e^{y_1(z, \varepsilon_n)}}{z^n} = \frac{1}{c_n},$$

from where we deduce that

$$\lim_{z\to\infty} \exp\left(\sum_{k=0}^{\infty} b_{1,k}(z)\varepsilon_n^{k-1} - n\ln z\right) = \frac{1}{c_n}. \tag{5.48}$$

From the relations $\varepsilon_n^M = \frac{1}{\lambda_n - \rho_{0,0}}$, $\lambda_n = \sum_{k=0}^{M} \rho_{k,k} \frac{n!}{(n-k)!}$ we obtain the algebraic equation

$$\sum_{m=1}^{M} \rho_{m,m} \frac{n!}{(n-m)!} - \frac{1}{\varepsilon_n^M} = 0. \tag{5.49}$$

From Lemma 5.6 applied to (5.49) gives that there exists an unique root n which depends analytically on the variable ε_n on some reduced neighborhood $V_\varepsilon^*(0)$ of $\varepsilon = 0$ such that

$$n = \sum_{k=0}^{\infty} h_k \varepsilon_n^{k-1}, \tag{5.50}$$

and $h_0 = 1$. It is not difficult to see from Lemma 5.6 that $h_1 = \frac{M-1}{2} - \frac{\rho_{M-1,M-1}}{M}$. Substituting (5.50) in (5.48) we get

$$\lim_{z\to\infty} \exp\left(\varepsilon_n^{-1}(b_{1,0}(z) - \ln z) + (b_{1,1}(z) - h_1 \ln z) + \sum_{k=2}^{\infty}(b_{1,k}(z) - h_k \ln z)\varepsilon_n^k\right) = \frac{1}{c_n}. \tag{5.51}$$

From the equations (5.17) of Lemma 5.3, we have that $b_{1,0}(z), b_{1,1}(z)$ for $z \in G_{\gamma,\mathfrak{r}_\zeta}$ are primitives of the functions $\dfrac{1}{\sqrt[M]{\rho_M(z)}}, \dfrac{M-1}{2M}\dfrac{\rho_M^{(1)}(z)}{\rho_M(z)} - \dfrac{1}{M}\dfrac{\rho_{M-1}(z)}{\rho_M(z)}$ respectively, where the branch of the function $\dfrac{1}{\sqrt[M]{\rho_M(z)}}$ is such that it coincides with $\dfrac{1}{z}$ at ∞.

We show now that it is possible to define the sequence of holomorphic functions $\{\Phi_k\}_{k=0}^{\infty}$ on $G_{\gamma,\mathfrak{r}_\zeta}$ from $\{b_{1,k}\}_{k=0}^{\infty}$ by adding an appropriate constant such that the limit in (5.48) equals to 1.

Let us choose Φ_0 as the primitive of $\dfrac{1}{\sqrt[M]{\rho_M(z)}}$ such that

$$\lim_{z\to\infty} \Phi_0(z) - \ln z = 0, \tag{5.52}$$

and Φ_1 as the primitive such that

$$\lim_{z\to\infty} \Phi_1(z) - h_1 \ln z = 0. \tag{5.53}$$

It is not difficult to see the existence and uniqueness of these primitives, which can be deduced from the Laurent expansion at ∞ of the respective integrands and taking into account (5.47).

Let us denote $m_k(z) = b_{1,k}(z) - h_k \ln z$. We show now that the limit $\lim\limits_{z\to\infty} m_k(z)$ exists for $k \geq 2$.

Indeed, notice that (5.51) implies that there exists $\eta > 0$ such that,

$$|m_k(z)| < \eta\varepsilon_n^{-k}, \quad z \in ext(\gamma), \tag{5.54}$$

for $k \geq k_0$, k_0 large enough.

Assume now that $1 < k < k_0$ and denote $r_k(z) = \operatorname{Re}[m_k(z)], s_k(z) = \Im[m_k(z)]$. From (5.51) we deduce that for all $z \in G_{\gamma,\mathfrak{r}_\zeta}$ there exist positive constants $\eta_1, \ldots, \eta_{2(k_0-2)}$ such that

$$\begin{aligned}
&-\eta_1 < (r_2(z)\operatorname{Re}[\varepsilon_n] - s_2(z)\Im[\varepsilon_n]) + \cdots + \left(r_{k_0-1}(z)\operatorname{Re}[\varepsilon_n^{k_0-2}] - s_{k_0-1}(z)\Im[\varepsilon_n^{k_0-2}]\right) < \eta_1 \\
&\qquad\vdots \\
&-\eta_{2(k_0-2)} < (r_2(z)\operatorname{Re}[\varepsilon_{n+2k_0-3}] - s_2(z)\Im[\varepsilon_{n+2k_0-3}]) + \ldots \\
&\qquad + \left(r_{k_0-1}(z)\operatorname{Re}[\varepsilon_{n+2k_0-3}^{k_0-2}] - s_{k_0-1}(z)\Im[\varepsilon_{n+k_0-3}^{k_0-2}]\right) < \eta_{2(k_0-2)}.
\end{aligned}$$

By reducing to diagonal form the above system of inequalities we get that there exists $N_1 > 0$ such that $|m_k(z)| < N_1, \forall z \in ext(\gamma)$, for $1 < k < k_0$. From this argument and (5.54) we obtain

$$|m_k(z)| < N\varepsilon_n^{-k}, \quad z \in ext(\gamma), \quad \forall k \geq 2, \tag{5.55}$$

for some $N > 0$.

Consider now a sequence $\{z_n\}_{n=0}^{\infty}$ of complex numbers such that $\lim\limits_{n\to\infty} z_n = \infty$. From (5.55) we deduce that $-\infty < \overline{\lim\limits_{z_n\to\infty}} \operatorname{Re}[m_k(z_n)] = a_k < \infty$ and $-\infty < \underline{\lim\limits_{z_n\to\infty}} \operatorname{Re}[m_k(z_n)] = a_k^{'} < \infty$, for $k \geq 2$. From (5.50) and (5.48) it holds

$$\sum_{k=2}^{\infty} (a_k - a_k^{'})\varepsilon_n^{k-1} = 0, \quad \forall n > n_0.$$

Hence, from [122, Th. 17.1, Vol. I] we have that $a_k - a_k^{'} = 0, \forall k \geq 2$, hence the existence of $\lim\limits_{z\to\infty} \operatorname{Re}[m_k(z)]$ for $k \geq 2$ is guaranteed. In a similar way, we obtain an analogous relation for the term $\Im[m_k(z)]$. Hence, $\lim\limits_{z\to\infty} m_k(z)$ exists for $k \geq 2$.

Let us consider $\Phi_k(z) = b_{1,k}(z) - m_k$, where $m_k = \lim\limits_{z\to\infty} m_k(z)$. Then

$$Q_n(z) = \exp\left(\sum_{k=0}^{\infty} \Phi_k(z)\varepsilon_n^{k-1}\right) \quad \forall z \in ext(\gamma). \tag{5.56}$$

Let $K \subset ext(\gamma)$ be a compact set. From (5.56) we deduce that $\{\Phi_k\}_{k=0}^{\infty}$ is uniformly bounded on K, hence

$$Q_n(z) = e^{\Phi_0(z)\varepsilon_n^{-1} + \Phi_1(z)} \left(1 + O(1/n)\right).$$

Notice that

$$\varepsilon_n^{-1} = \sqrt[M]{\lambda_n - \rho_{0,0}} = n - \left(\frac{M-1}{2} - \frac{\rho_{M-1,M-1}}{M}\right) + O(1/n).$$

Hence,

$$Q_n(z) = e^{\left(n - \left(\frac{M-1}{2} - \frac{\rho_{M-1,M-1}}{M}\right)\right)\Phi_0(z) + \Phi_1(z)} \left(1 + O(1/n)\right),$$

which is a). Relations b) and c) follows immediately from a).

□

5.4 Applications.

In [89] the authors find an interesting example of a linear differential operator of order four

$$\mathcal{L}^{(M)}[u] = (z^2-1)^2 u^{(4)} + 4z(z^2-1)u^{(3)} + 2(z-1)((1+2\,c)z + 2\,c + 3)u^{(2)},$$

where the sequence polynomial eigenfunctions $\{Q_n\}_{n=0}^{\infty}$ with eigenvalues $\lambda_n = n(n-1)(n^2 - n + 4\,c)$, are orthogonal with respect to the inner product

$$\langle P, Q\rangle = P(1)\overline{Q}(1) + \frac{1}{c}\,P'(1)\overline{Q}'(1) + \int_{-1}^{1} P'\overline{Q}'\,dx, \quad P, Q \in \mathbb{P}, \quad c > 0.$$

The results of the preceding section can be applied to study the strong asymptotic behavior of the sequence of monic orthogonal polynomials with respect to this inner product. From [15] we deduce that the zero counting measure of these polynomial converges in the $*$–weak topology to a measure ν with support $[-1,1]$. Consider a closed Jordan curve γ which encloses $[-1,1]$. From Theorem 5.2 we obtain

$$\Phi_0(z) = \int \frac{d\zeta}{\sqrt[M]{\rho_M(\zeta)}} = \ln\left[2\left(z + \sqrt{z^2-1}\right)\right] + c, \quad z \in G_{\gamma,\mathfrak{r}_\zeta}.$$

From the relations (5.47), (5.52)

$$\Phi_0(z) = \ln\left[2\left(z + \sqrt{z^2-1}\right)\right] - \ln 4, \quad z \in ext(\gamma),$$

where we take the branch of $\sqrt{z^2-1}$ for which $|\varphi(z)| > 1$ whenever $z \in \mathbb{C}\setminus[-1,1]$. In a similar way,

$$\begin{aligned}\Phi_1(z) &= \int \left(\frac{(M-1)\rho_M^{'}(\zeta)}{2M\rho_M(\zeta)} - \frac{\rho_{M-1}(\zeta)}{M\rho_M(\zeta)} \right) d\zeta \\ &= \frac{1}{4} \ln\left(z^2-1\right) + c, \quad z \in G_{\gamma,\mathfrak{r}_\zeta}.\end{aligned}$$

From (5.47), (5.53) we deduce that $\Phi_1(z) = \frac{1}{4}\ln(z^2-1), z \in ext(\gamma)$.
Therefore,

a) $Q_n(z) = \left(\frac{\varphi(z)}{2}\right)^{n-1/2} \sqrt[4]{z^2-1}\,(1+O(1/n))$,

b) $\lim\limits_{n\to\infty} \frac{Q_{n+1}(z)}{Q_n(z)} = \frac{\varphi(z)}{2}$,

c) $\lim\limits_{n\to\infty} \sqrt[n]{|Q_n(z)|} = \frac{|\varphi(z)|}{2}$,

uniformly on $K \subset ext(\gamma)$. Here we choose the branch of the roots $\sqrt[4]{z}, \sqrt{z}$ from the conditions $\sqrt[4]{1} = 1, \sqrt{1} = 1$.

Chapter 6

Orthogonal matrix polynomials satisfying differential equations with recurrence coefficients having non-scalar limits

6.1 Introduction

The purpose of this chapter is to introduce a new family of weight matrices W of the form TT^*, $T(t) = e^{\mathscr{A}t}e^{\mathscr{D}t^2}$, where $\mathscr{A}$ is certain nilpotent matrix and $\mathscr{D}$ is a diagonal matrix with negative real entries. The weight matrices W have arbitrary size $N \times N$ and depend on N parameters. The orthogonal polynomials with respect to this family of weight matrices are eigenfunctions of a second order differential operator

$$\left(\frac{d}{dt}\right)^2 F_2(t) + \left(\frac{d}{dt}\right)^1 F_1(t) + F_0(t), \tag{6.1}$$

whose coefficients are matrix polynomials F_2, F_1 and F_0 (independent of n) of degrees not bigger than 2, 1 and 0 respectively.

If $\{P_n\}_{n=0}^{\infty}$ is a sequence of orthonormal matrix polynomials with respect to W, the symmetry of the second order differential operator (6.1) is equivalent to saying that P_n satisfy the second order differential equation

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0(t) = \Lambda_n P_n(t), \tag{6.2}$$

where Λ_n are Hermitian matrices (see [45, Lemma 4]). As usual, the symmetry of an operator D with respect to the weight matrix W is defined by $\int D[P]dWQ^* = \int PdW(D[Q])^*$, for any matrix polynomials P, Q.

For size 2×2, we find an explicit expression for a sequence of orthonormal polynomials with respect to W. In particular, we show that one of the recurrence coefficients for this sequence of orthonormal polynomials does not asymptotically behave as a scalar multiple of the identity, as it happens in the examples studied up to now in the literature.

Our weight matrices are of arbitrary size $N \times N$ and are constructed from the $N-1$ non zero complex parameters $a_1, \dots, a_{N-1}$ and the positive real parameter b ($b \neq 1$) as follows. Consider the nilpotent matrix A

and the diagonal matrices $\mathscr{J}$ and Ψ defined by

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \mathscr{J} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N-1 \end{pmatrix}, \tag{6.3}$$

$$\Psi = I + \frac{b-1}{N-1}\mathscr{J}. \tag{6.4}$$

Let the diagonal matrix $\mathscr{D}$ and the upper triangular nilpotent matrix $\mathscr{A}$ be defined by

$$\mathscr{D} = -\frac{b}{2}\Psi^{-1}, \tag{6.5}$$

$$\mathscr{A} = \sum_{j=0}^{[\frac{N}{2}]-1} \alpha_j A^{2j+1}, \tag{6.6}$$

where

$$\alpha_j = \frac{(1-b)^j(2j+1)^{j-1}}{(4b)^j(N-1)^j j!}, \quad j \geq 0.$$

The weight matrix W is then defined by

$$W(t) = T(t)T^*(t), \quad T(t) = e^{\mathscr{A}t}e^{\mathscr{D}t^2}. \tag{6.7}$$

Since $\mathscr{A}$ is nilpotent of order N, $e^{\mathscr{A}t}$ is always a polynomial of degree $N-1$.

For $b = 1$ (considering $\alpha_0 = 1$), we recover [49, Example 5.1].

For the benefit of the reader, we display here our weight matrix for size 2×2. For $N = 2$ we have

$$\mathscr{A} = A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \mathscr{D} = \begin{pmatrix} -\frac{b}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

and then

$$W = \begin{pmatrix} |a|^2t^2e^{-t^2} + e^{-bt^2} & ate^{-t^2} \\ \overline{a}te^{-t^2} & e^{-t^2} \end{pmatrix}, \quad T = \begin{pmatrix} e^{-bt^2/2} & ate^{-t^2/2} \\ 0 & e^{-t^2/2} \end{pmatrix}. \tag{6.8}$$

In Section 6.2, we prove that our weight matrix W always has a symmetric second order differential operator like (6.1):

THEOREM 6.1. *The second order differential operator (6.1) with differential coefficients F_2, F_1 and F_0 given by*

$$F_2(t) = \Psi + \frac{b-1}{N-1}[\mathscr{A}, \mathscr{J}]\,t, \tag{6.9}$$

$$F_1(t) = 2\mathscr{A}\Psi + 2\left(-bI + \frac{b-1}{N-1}\mathscr{A}[\mathscr{A}, \mathscr{J}]\right)t, \tag{6.10}$$

$$F_0(t) = 2b\mathscr{J} + \mathscr{A}^2\Psi, \tag{6.11}$$

is symmetric with respect to the weight matrix W (6.7) (as usual $[X,Y]$ denotes the commutator of the matrices X, Y: $[X,Y] = XY - YX$).

For $N = 2$, these differential coefficients are

$$F_2(t) = \begin{pmatrix} 1 & a(b-1)t \\ 0 & b \end{pmatrix}, \quad F_1(t) = \begin{pmatrix} -2bt & 2ab \\ 0 & -2bt \end{pmatrix}, \quad F_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 2b \end{pmatrix}.$$

The rest of the sections are devoted to study in depth the orthogonal polynomials with respect to our weight matrix for size 2×2. The study of the orthogonal polynomials for higher size N, $N \geq 3$, remains a challenge.

In Section 6.3, we construct the following Rodrigues' formula for a sequence of orthogonal polynomials with respect to the weight matrix W given by (6.8):

THEOREM 6.2. *Let the function P_n, $n \geq 1$, be defined by*

$$P_n(t) = (-1)^n \left[e^{-t^2} \begin{pmatrix} b^{-n} e^{(1-b)t^2} + \dfrac{|a|^2}{2}(n + 2t^2) & at \\ \bar{a} \left[2t + e^{t^2} \sqrt{\pi} n \Big(\operatorname{Erf}(\sqrt{b}t) - \operatorname{Erf}(t) \Big) \right] & 2 \end{pmatrix} \right]^{(n)} W^{-1}, \tag{6.12}$$

where Erf *denotes the error function* $\operatorname{Erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-x^2} dx$. *Then P_n, $n \geq 1$, is a polynomial of degree n with nonsingular leading coefficient*

$$\Gamma_n = 2^n \begin{pmatrix} 1 & 0 \\ 0 & \gamma_n \end{pmatrix}, \quad \gamma_n = 2 + |a|^2 b^{n-\frac{1}{2}} n. \tag{6.13}$$

Moreover, defining $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, *$\{P_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to W.*

To find the Rodrigues' formula, we will apply Theorem 1.11. We will also make use of the following well known formula: for any matrices $X, Y \in \mathbb{C}^{N \times N}$:

$$e^X Y = \left(\sum_{n \geq 0} \frac{t^n}{n!} \operatorname{ad}_X^n Y \right) e^X, \tag{6.14}$$

where we use the standard notation

$$\operatorname{ad}_X^0 Y = Y, \quad \operatorname{ad}_X^1 Y = [X, Y] = XY - YX, \quad \operatorname{ad}_X^2 Y = [X, [X, Y]],$$

and in general, $\operatorname{ad}_X^{n+1} Y = [X, [\operatorname{ad}_X^n Y]]$.

The Rodrigues' formula allows us to find an explicit expression for the polynomials $\{P_n\}_{n=0}^{\infty}$ in terms of the Hermite polynomials.

COROLLARY 6.1. *For $n \geq 1$, we have*

$$P_n(t) = \begin{pmatrix} b^{-n/2} H_n(\sqrt{b}t) & -atb^{-n/2} H_n(\sqrt{b}t) + \frac{a}{2} H_{n+1}(t) \\ -2\bar{a} b^{n/2} n H_{n-1}(\sqrt{b}t) & 2|a|^2 b^{n/2} nt H_{n-1}(\sqrt{b}t) + 2H_n(t) \end{pmatrix}, \tag{6.15}$$

where H_n is the n-th Hermite polynomial defined by $H_n(t) = (-1)^n \left(e^{-t^2} \right)^{(n)} e^{t^2}$.

In Section 6.4, using again the Rodrigues' formula (6.12), we find the following three term recurrence relation for a sequence $\{\mathscr{P}_n\}_{n=0}^{\infty}$ of orthonormal polynomials with respect to W.

THEOREM 6.3. *The sequence of matrix polynomials defined by* $\mathscr{P}_{-1} = 0$, $\mathscr{P}_0 = (\pi)^{-\frac{1}{4}} \begin{pmatrix} \sqrt{\frac{2\sqrt{b}}{\gamma_1}} & 0 \\ 0 & 1 \end{pmatrix}$

and

$$t\mathscr{P}_n(t) = A_{n+1}\mathscr{P}_{n+1}(t) + B_n\mathscr{P}_n(t) + A_n^*\mathscr{P}_{n-1}(t), \quad n \geq 0, \tag{6.16}$$

where

$$A_n = \sqrt{n} \begin{pmatrix} \sqrt{\frac{\gamma_{n+1}}{2b\gamma_n}} & 0 \\ 0 & \sqrt{\frac{\gamma_{n-1}}{2\gamma_n}} \end{pmatrix}, \tag{6.17}$$

$$B_n = \frac{b^{\frac{2n-3}{4}}(b+(b-1)n)}{\sqrt{\gamma_n\gamma_{n+1}}} \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}, \tag{6.18}$$

is orthonormal with respect to W *(6.8) (where the sequence* $\{\gamma_n\}_{n=0}^{\infty}$ *is defined by (6.13)).*

This gives for the recurrence coefficients $\{A_n\}_{n=0}^{\infty}$ the asymptotic behavior

$$\lim_{n\to\infty} \frac{A_n}{\sqrt{n}} = \begin{cases} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2b}} \end{pmatrix} & \text{if } b > 1 \\ \begin{pmatrix} \frac{1}{\sqrt{2b}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} & \text{for } 0 < b < 1 \end{cases}.$$

This limit shows that the recurrence coefficients $\{A_n\}_{n=0}^{\infty}$ do not asymptotically behave as a scalar multiple of the identity, as it happens in the examples studied up to now in the literature.

6.2 Symmetric differential operator

In this section we prove Theorem 6.1, that is, the second order differential operator with coefficients given by (6.9), (6.10) and (6.11) is symmetric with respect to the weight matrix W (6.7).

We now list some technical relations which we will need in the proof of Theorem 6.1 (they will be proved later).

LEMMA 6.1. *Let the function* F_2 *and the matrices* A, $\mathscr{A}$, Ψ, $\mathscr{D}$ *and* $\mathscr{J}$ *be defined by (6.9), (6.3), (6.4), (6.5)* *and (6.6), respectively. Then*

$$[\mathscr{A},\mathscr{J}] = \sum_{j=0}^{[\frac{N}{2}]-1} (2j+1)\alpha_j A^{2j+1}. \tag{6.19}$$

$$e^{\mathscr{A}t}\Psi = F_2 e^{\mathscr{A}t}. \tag{6.20}$$

$$e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}\Psi = -\frac{b}{2}I - \frac{(b-1)t}{N-1}e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}[\mathscr{A},\mathscr{J}]. \tag{6.21}$$

$$\Psi e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t} = -\frac{b}{2}I - \frac{(b-1)t}{N-1}[\mathscr{A},\mathscr{J}]e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}. \tag{6.22}$$

$$\mathscr{A}\,[\mathscr{A},\mathscr{J}] = \frac{2b(N-1)}{1-b}\sum_{j=1}^{[\frac{N-1}{2}]}\alpha_j\frac{(2j)^j}{(2j+1)^{j-1}}A^{2j}. \tag{6.23}$$

$$[\mathscr{A},\mathscr{J}] - \mathscr{A} = \frac{(1-b)}{2b(N-1)}\mathscr{A}^2\,[\mathscr{A},\mathscr{J}]\,. \tag{6.24}$$

We are now ready to prove Theorem 6.1.

Proof. (of Theorem 6.1)

The symmetry of the second order differential operator with respect to the weight matrix W will be a consequence of Theorems 1.9 and 1.10. We have to check the boundary conditions (1.17) and the three equations (1.15) and (1.16).

To make the proof easier to follow, we proceed in four steps.

First step: Boundary conditions (1.17). Proof: Since $\mathscr{A}$ is nilpotent, we deduce that $e^{\mathscr{A}t}$ is a polynomial. The matrix function $T(t) = e^{\mathscr{A}t}e^{\mathscr{D}t^2}$ then decays exponentially at ∞ because the entries of the diagonal matrix $\mathscr{D}$ are negative. Hence the weight matrix $W = TT^*$ also decays exponentially at ∞. Since F_2 and F_1 are polynomials, it follows straightforwardly that $t^n F_2 W$ and $t^n[(F_2W)' - F_1W]$, $n \geq 0$, have vanishing limits at ∞.

Second step: $F_2W = WF_2^*$. Proof: Formula (6.20) of Lemma 6.1 shows that $F_2T = T\Psi$, where Ψ is the diagonal matrix (real entries) defined by (6.4) and $T = e^{\mathscr{A}t}e^{\mathscr{D}t^2}$. Then

$$F_2TT^* = T\Psi T^* = T(T\Psi)^* = T(F_2T)^* = TT^*F_2^*.$$

Since $W = TT^*$, we get that $F_2W = WF_2^*$.

Third step: $2(F_2W)' = F_1W + WF_1^*$. Proof: To check the equation $2(F_2W)' = F_1W + WF_1^*$, we use the first part of Theorem 1.10 with $\Omega = \mathbb{R}$. Hence, we have to prove that T satisfies $T'(t) = F(t)T(t)$, where F is a solution of the matrix equation

$$F_1(t) = F_2(t)F(t) + F(t)F_2(t) + F_2'(t). \tag{6.25}$$

Taking into account that $T = e^{\mathscr{A}t}e^{\mathscr{D}t^2}$, a direct computation gives that

$$F(t) = \mathscr{A} + 2te^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}. \tag{6.26}$$

The definition of F_2 (6.9) and (6.26) give

$$\begin{aligned}F_2F + FF_2 = \mathscr{A}\Psi + \Psi\mathscr{A} + \frac{2(b-1)t}{N-1}\mathscr{A}[\mathscr{A},\mathscr{J}] + 2t(\Psi e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t} + e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}\Psi)\\ + \frac{2(b-1)t^2}{N-1}(e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}[\mathscr{A},\mathscr{J}] + [\mathscr{A},\mathscr{J}]e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}).\end{aligned}$$

Using the definition of Ψ (6.4), (6.21) and (6.22) of Lemma 6.1 we get

$$F_2F + FF_2 = 2\mathscr{A} + \frac{(b-1)}{N-1}(\mathscr{J}\mathscr{A} + \mathscr{A}\mathscr{J}) + \frac{2t(b-1)}{N-1}\mathscr{A}[\mathscr{A}, \mathscr{J}] - 2bIt.$$

Formula (6.25) now follows easily taking into account the definitions of F_2 (6.9), F_1 (6.10) and Ψ (6.4).

Fourth step: $(F_2W)'' - (F_1W)' + F_0W = WF_0^*$. Proof:
Using 2 of Theorem 1.10, this is equivalent to prove that the matrix

$$\chi = T^{-1}(-FF_2F - (FF_2)' + F_0)T \tag{6.27}$$

is Hermitian.

We actually will prove that the matrix function χ defined by (6.27) is diagonal with real entries.

Taking into account that $T(t) = e^{\mathscr{A}t}e^{\mathscr{D}t^2}$, and $\mathscr{D}$ is diagonal, it is enough to prove that the matrix function

$$\xi = e^{-\mathscr{A}t}(-FF_2F - (FF_2)' + F_0)e^{\mathscr{A}t} \tag{6.28}$$

is diagonal.

We first compute $e^{-\mathscr{A}t}(FF_2F)e^{\mathscr{A}t}$.

Taking into account the expression for $F(t)$ in (6.26), that $\mathscr{A}$ and $e^{\mathscr{A}t}$ commute and using (6.20), one has after straightforward computations

$$e^{-\mathscr{A}t}(FF_2F)e^{\mathscr{A}t} = \mathscr{A}\Psi\mathscr{A} + 2t(\mathscr{A}\Psi\mathscr{D} + \mathscr{D}\Psi\mathscr{A}) + 4t^2\mathscr{D}\Psi\mathscr{D}.$$

The definition of $\mathscr{D}$ (6.5) and Ψ (6.4) now give

$$e^{-\mathscr{A}t}(FF_2F)e^{\mathscr{A}t} = \mathscr{A}^2 + \frac{b-1}{N-1}\mathscr{A}\mathscr{J}\mathscr{A} - 2bt\mathscr{A} - 2t^2b\mathscr{D}. \tag{6.29}$$

We now compute $e^{-\mathscr{A}t}(FF_2)'e^{\mathscr{A}t}$. Using again the definition of F (6.26) and (6.20) of Lemma 6.1, we have that

$$(FF_2)' = (\mathscr{A}F_2 + 2te^{\mathscr{A}t}\mathscr{D}\Psi e^{-\mathscr{A}t})'.$$

The definition of $\mathscr{D}$ (6.5) and F_2 (6.9) give

$$\begin{aligned}(FF_2)' &= \left(\mathscr{A}\Psi + \frac{(b-1)t}{N-1}\mathscr{A}[\mathscr{A}, \mathscr{J}] - btI\right)' \\ &= \frac{b-1}{N-1}\mathscr{A}[\mathscr{A}, \mathscr{J}] - bI.\end{aligned}$$

Identity (6.19) of Lemma 6.1 shows that $e^{\mathscr{A}t}$ and $[\mathscr{A}, \mathscr{J}]$ commute (they are linear combinations of power of A). One then obtains

$$e^{-\mathscr{A}t}(FF_2)'e^{\mathscr{A}t} = \frac{b-1}{N-1}\mathscr{A}[\mathscr{A}, \mathscr{J}] - bI. \tag{6.30}$$

We finally compute $e^{-\mathscr{A}t}F_0e^{\mathscr{A}t}$. The definition of F_0 (6.11) gives

$$\begin{aligned}e^{-\mathscr{A}t}F_0e^{\mathscr{A}t} &= e^{-\mathscr{A}t}(2b\mathscr{J} + \mathscr{A}^2\Psi)e^{\mathscr{A}t} \\ &= 2be^{-\mathscr{A}t}\mathscr{J}e^{\mathscr{A}t} + \mathscr{A}^2e^{-\mathscr{A}t}\Psi e^{\mathscr{A}t}.\end{aligned}$$

Again (6.19) of Lemma 6.1 shows that $\mathscr{A}$ and $[\mathscr{A}, \mathscr{J}]$ commute. Hence $\mathrm{ad}^n_{\mathscr{A}}\mathscr{J} = 0$, $n \geq 2$. Using this fact and (6.14) one obtains

$$e^{-\mathscr{A}t}F_0e^{\mathscr{A}t} = 2b(\mathscr{J} - [\mathscr{A}, \mathscr{J}]t) + \mathscr{A}^2(\Psi - [\mathscr{A}, \Psi]t).$$

The definition of Ψ (6.4) gives

$$e^{-\mathscr{A}t}F_0e^{\mathscr{A}t} = 2b\mathscr{J} - 2bt[\mathscr{A},\mathscr{J}] + \mathscr{A}^2 + \frac{b-1}{N-1}\mathscr{A}^2\mathscr{J} - \frac{b-1}{N-1}\mathscr{A}^2[\mathscr{A},\mathscr{J}]t. \tag{6.31}$$

We now substitute (6.29), (6.30) and (6.31) in the definition of ξ (6.28) obtaining

$$\begin{aligned}\xi = {}& \frac{b-1}{N-1}(-\mathscr{A}\mathscr{J}\mathscr{A} - \mathscr{A}[\mathscr{A},\mathscr{J}] + \mathscr{A}^2\mathscr{J}) + 2bt(\mathscr{A} - [\mathscr{A},\mathscr{J}] \\ & - \frac{b-1}{2b(N-1)}\mathscr{A}^2[\mathscr{A},\mathscr{J}]) + 2bt^2\mathscr{D} + bI + 2b\mathscr{J},\end{aligned}$$

(6.24) of lemma 6.1 finally gives

$$\xi = bI + 2bt^2\mathscr{D} + 2b\mathscr{J},$$

which it is indeed a diagonal matrix.

This concludes the proof of Theorem 1.1.

□

It remains to prove Lemma 6.1.

Proof. (of Lemma 6.1)

First step. Proof of (6.19):

Using induction on k, we easily find that $[A^k,\mathscr{J}] = kA$, $k \geq 1$. This shows that $\operatorname{ad}_A^n \mathscr{J} = 0$, $n \geq 2$. The definition of $\mathscr{A}$ (6.6) gives now (6.19).

Second step. Proof of (6.20), (6.21) and (6.22):

From (6.4)

$$[\mathscr{A},\Psi] = \frac{b-1}{N-1}[\mathscr{A},\mathscr{J}]. \tag{6.32}$$

From $\mathscr{A}$ (6.6) and (6.19) the matrices $\mathscr{A}$, $[\mathscr{A},\mathscr{J}]$ and $e^{\mathscr{A}t}$ commute (they are linear combination of powers of A). We then have

$$\operatorname{ad}_{\mathscr{A}}^n(\Psi) = 0, \quad n \geq 2, \tag{6.33}$$

$$[\mathscr{A},\mathscr{J}]e^{\mathscr{A}t} = e^{\mathscr{A}t}[\mathscr{A},\mathscr{J}]. \tag{6.34}$$

Using (6.14) and (6.33) we get

$$e^{\mathscr{A}t}\Psi e^{-\mathscr{A}t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\operatorname{ad}_{\mathscr{A}}^n\Psi = \Psi + t[\mathscr{A},\Psi].$$

From (6.4) and F_2 (6.9) we get (6.20).

In a similar way, we have

$$e^{-\mathscr{A}t}\Psi e^{\mathscr{A}t} = \Psi - t[\mathscr{A},\Psi].$$

Using now the definition of $\mathscr{D}$ (6.5), (6.32) and (6.34) we have

$$\begin{aligned}e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}\Psi &= e^{\mathscr{A}t}\mathscr{D}(\Psi - t[\mathscr{A},\Psi])e^{-\mathscr{A}t} \\ &= -\frac{b}{2}I - \frac{(b-1)t}{N-1}e^{\mathscr{A}t}\mathscr{D}e^{-\mathscr{A}t}[\mathscr{A},\mathscr{J}].\end{aligned}$$

This proves (6.21). The proof of (6.22) is similar.

Third step. Proof of (6.23): From (6.6) and (6.19), we must prove the following identity

$$\sum_{j=0}^{[\frac{N}{2}]-1} \alpha_j A^{2j+1} \sum_{j=0}^{[\frac{N}{2}]-1} (2j+1)\alpha_j A^{2j+1} = \frac{2b(N-1)}{1-b} \sum_{j=1}^{[\frac{N-1}{2}]} \alpha_j \frac{(2j)^j}{(2j+1)^{j-1}} A^{2j}.$$

This is equivalent to prove

$$\sum_{j,s=0}^{[\frac{N}{2}]-1} \alpha_j \alpha_s (2s+1) A^{2(j+s+1)} = \frac{2b(N-1)}{1-b} \sum_{j=1}^{[\frac{N-1}{2}]} \alpha_j \frac{(2j)^j}{(2j+1)^{j-1}} A^{2j}. \tag{6.35}$$

Taking into account that A is nilpotent,

$$\alpha_s \alpha_j = \alpha_{j+s} \frac{(2j+1)^{j-1}(2s+1)^{s-1}}{(2(j+s)+1)^{j+s-1}} \left(\begin{array}{c} s+j \\ s \end{array} \right), \tag{6.36}$$

and writing $k = j+s$, we find

$$\sum_{j,s=0}^{[\frac{N}{2}]-1} \alpha_j \alpha_s (2s+1) A^{2(j+s+1)} \tag{6.37}$$

$$= \sum_{k=0}^{[\frac{N-1}{2}]-1} \frac{\alpha_k}{(2k+1)^{k-1}} A^{2(k+1)} \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) (2m+1)^m (2(k-m)+1)^{k-m-1}.$$

We now use Abel's binomial identity (see for instance [148, p. 18]): for $z, w \in \mathbb{C}$, $w \neq 0$,

$$\sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) (m+z)^m (k-m+w)^{k-m-1} = w^{-1}(z+w+k)^k. \tag{6.38}$$

Then

$$\sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) (2m+1)^m (2(k-m)+1)^{k-m-1} = 2^k (1+k)^k,$$

(6.37) now gives

$$\sum_{j,s=0}^{[\frac{N}{2}]-1} \alpha_j \alpha_s (2s+1) A^{2(j+s+1)} = \sum_{j=0}^{[\frac{N-1}{2}]-1} \alpha_j \frac{2^j (j+1)^j}{(2j+1)^{j-1}} A^{2(j+1)}.$$

Using (6.36) one has

$$\alpha_j \alpha_1 = \frac{\alpha_{j+1}(j+1)(2j+1)^{j-1}}{(2j+3)^j}, \qquad \text{with } \alpha_1 = \frac{1-b}{4b(N-1)}.$$

Thus,

$$\sum_{j=0}^{[\frac{N-1}{2}]-1} \alpha_j \frac{2^j (j+1)^j}{(2j+1)^{j-1}} A^{2(j+1)} = \frac{2b(N-1)}{1-b} \sum_{j=0}^{[\frac{N-1}{2}]-1} \alpha_{j+1} \frac{2^{j+1}(j+1)^{j+1}}{(2j+3)^j} A^{2(j+1)}.$$

This proves (6.35) and then (6.23) as well.

Four step. Proof of (6.24): Taking into account (6.19) and (6.23), we have to prove the following identity

$$\sum_{j=1}^{[\frac{N}{2}]-1} 2j\alpha_j A^{2j+1} = \sum_{j=0}^{[\frac{N}{2}]-1} \alpha_j A^{2j+1} \sum_{j=1}^{[\frac{N-1}{2}]} \alpha_j \frac{(2j)^j}{(2j+1)^{j-1}} A^{2j}.$$

This is equivalent to prove

$$\sum_{j=1}^{[\frac{N}{2}]-1} 2j\alpha_j A^{2j+1} = \sum_{s=0}^{[\frac{N}{2}]-1} \sum_{j=1}^{[\frac{N-1}{2}]} \alpha_s \alpha_j \frac{(2j)^j}{(2j+1)^{j-1}} A^{2(s+j)+1}. \tag{6.39}$$

Using (6.36) once again, we have

$$\begin{aligned} &\sum_{s=0}^{[\frac{N}{2}]-1} \sum_{j=1}^{[\frac{N-1}{2}]} \alpha_s \alpha_j \frac{(2j)^j}{(2j+1)^{j-1}} A^{2(s+j)+1} \\ &\quad = \sum_{s=0}^{[\frac{N}{2}]-1} \sum_{j=1}^{[\frac{N-1}{2}]} \alpha_{s+j} \frac{(2j)^j (2s+1)^{s-1}}{(2(j+s)+1)^{j+s-1}} \begin{pmatrix} s+j \\ s \end{pmatrix} A^{2(s+j)+1}. \end{aligned}$$

Writing $k = s + j$, and taking into account that A is nilpotent of order N, we get for the right hand side of (6.39) the expression

$$\sum_{k=1}^{[\frac{N}{2}]-1} \frac{\alpha_k}{(2k+1)^{k-1}} A^{2k+1} \sum_{j=1}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (2j)^j (2(k-j)+1)^{k-j-1}.$$

Writing $m = k - j$, one obtains

$$\sum_{k=1}^{[\frac{N}{2}]-1} \frac{2^{k-1}\alpha_k}{(2k+1)^{k-1}} A^{2k+1} \sum_{m=0}^{k-1} \begin{pmatrix} k \\ m \end{pmatrix} (k-m)^{k-m} (m+\frac{1}{2})^{m-1}.$$

Abel's binomial identity (6.38) now gives

$$\sum_{m=0}^{k-1} \begin{pmatrix} k \\ m \end{pmatrix} (k-m)^{k-m} (m+\frac{1}{2})^{m-1} = \frac{1}{2^{k-1}} \left((2k+1)^k - (2k+1)^{k-1}\right).$$

From where one can easily deduce (6.39).

This proves (6.24).

The proof of the lemma is now complete.

□

6.3 Rodrigues Formula

In this section we will prove Theorem 6.2 which provides a Rodrigues' Formula for a sequence of orthogonal polynomials with respect to the weight matrix W for size 2×2 (6.8).

Let us write

$$R_n = (-1)^n e^{-t^2} \begin{pmatrix} b^{-n} e^{(1-b)t^2} + \frac{|a|^2}{2}(n+2t^2) & at \\ \bar{a}\left[2t + e^{t^2}\sqrt{\pi}n\left(\mathrm{Erf}(\sqrt{b}t) - \mathrm{Erf}(t)\right)\right] & 2 \end{pmatrix}, \tag{6.40}$$

where, as usual, Erf denotes the error function $\mathrm{Erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-x^2} dx$.

The Rodrigues' formula (6.12) can then be written as $P_n = R_n^{(n)} W^{-1}$.

First of all, we explain how one can use Theorem 1.11 to find these functions R_n, $n \geq 1$.

Indeed, Theorem 6.1 for size 2×2 gives for the weight matrix W the following symmetric second order differential operator

$$D = \left(\frac{d}{dt}\right)^2 \begin{pmatrix} 1 & a(b-1)t \\ 0 & b \end{pmatrix} + \left(\frac{d}{dt}\right) \begin{pmatrix} -2bt & 2ab \\ 0 & -2bt \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2b \end{pmatrix}. \tag{6.41}$$

Since the operator D is symmetric with respect to W, the n-th monic orthogonal polynomial $\hat{P}_n$ with respect to W satisfies the differential equation $D(\hat{P}_n) = \Lambda_n \hat{P}_n$, where the eigenvalues Λ_n are given by

$$\Lambda_n = \begin{pmatrix} -2bn & 0 \\ 0 & -2b(n-1) \end{pmatrix}.$$

Theorem 1.11 associates the following second order differential equation (1.27) to the differential operator D (6.41):

$$\left[R_n \begin{pmatrix} 1 & 0 \\ a(b-1)t & b \end{pmatrix}\right]'' - \left[R_n \begin{pmatrix} -2bt & 0 \\ a[b(2+n)-n] & -2bt \end{pmatrix}\right]' + R_n\Lambda_n = \Lambda_n R_n. \tag{6.42}$$

Take now a solution R_n of this differential equation and write $Y_n = R_n^{(n)} W^{-1}$. Theorem 1.11 guarantees that the function Y_n satisfies the differential equation $D(Y_n) = \Lambda_n Y_n$. Notice that Y_n and $\hat{P}_n$ satisfy the same differential equation. We have hence looked for a solution R_n of the differential equation (6.42) such that the matrix function $R_n^{(n)} W^{-1}$ is also a matrix polynomial of degree n with nonsingular leading coefficient. This is the procedure we have used to find the functions R_n given by (6.40).

We now prove Theorem 6.2, which establishes that actually the functions $R_n^{(n)} W^{-1}$ define a sequence of orthogonal polynomials with respect to W.

Proof. (of Theorem 6.2)

Using the Rodrigues' formula for Hermite polynomials, $H_n(t) = (-1)^n \left(e^{-t^2}\right)^{(n)} e^{t^2}$, [163, Chapter 5], we have that

$$\begin{aligned}
\left(t^2 e^{-t^2}\right)^{(n)} &= \frac{(-1)^n}{4} H_{n+2}(t) e^{-t^2} + \frac{(-1)^n}{2} H_n(t) e^{-t^2}, \\
\left(t e^{-t^2}\right)^{(n)} &= \frac{(-1)^n}{2} H_{n+1}(t) e^{-t^2}, \\
\left(e^{-bt^2}\right)^{(n)} &= (-1)^n (\sqrt{b})^n H_n(\sqrt{b}t) e^{-bt^2}, \\
(\mathrm{Erf}(t))^{(n)} &= (-1)^{n-1} \frac{2}{\sqrt{\pi}} H_{n-1}(t) e^{-t^2}, \\
(\mathrm{Erf}(\sqrt{b}t))^{(n)} &= (-1)^{n-1} b^{n/2} \frac{2}{\sqrt{\pi}} H_{n-1}(\sqrt{b}t) e^{-bt^2}.
\end{aligned}$$

These identities give, after straightforward computations using the three term recurrence relation for the Hermite polynomials $tH_n = H_{n+1}/2 + nH_{n-1}$ (see [163, Chapter 5]):

$$R_n^{(n)}(t) = \begin{pmatrix} b^{-n/2} H_n(\sqrt{b}t) e^{-bt^2} + \dfrac{|a|^2}{2} t H_{n+1}(t) e^{-t^2} & \dfrac{a}{2} H_{n+1}(t) e^{-t^2} \\ 2\overline{a}\left(t H_n(t) e^{-t^2} - n b^{n/2} H_{n-1}(\sqrt{b}t) e^{-bt^2}\right) & 2H_n(t) e^{-t^2} \end{pmatrix}.$$

Taking into account that

$$W^{-1} = \begin{pmatrix} e^{bt^2} & -ae^{bt^2}t \\ -\overline{a}e^{bt^2}t & |a|^2e^{bt^2}t^2 + e^{t^2} \end{pmatrix},$$

we finally find the expression (6.15) for P_n given in Corollary 6.1:

$$R_n^{(n)}(t)W^{-1} = \begin{pmatrix} b^{-n/2}H_n(\sqrt{b}t) & -atb^{-n/2}H_n(\sqrt{b}t) + \frac{a}{2}H_{n+1}(t) \\ -2\overline{a}b^{n/2}nH_{n-1}(\sqrt{b}t) & 2|a|^2b^{n/2}ntH_{n-1}(\sqrt{b}t) + 2H_n(t) \end{pmatrix}.$$

This shows that $R_n^{(n)}W^{-1}$ is a polynomial of degree n (note that the entry $(1,2)$ of this matrix is actually a polynomial of degree $n-1$) with nonsingular leading coefficient equal to Γ_n (6.13).

The orthogonality of P_n and t^kI, $0 \leq k \leq n-1$, with respect to W follows taking into account that

$$\int P_n(t)W(t)t^k dt = \int R_n^{(n)}(t)t^k dt,$$

and performing a careful integration by parts.

□

6.4 Three term recurrence relation

In order to find the recurrence coefficients (6.17) in the three term recurrence relation of Theorem 6.3 we have followed the strategy of [54] or [56].

Proof. (of Theorem 6.3)

We first compute the L^2 norm of the monic orthogonal polynomials $\hat{P}_n$ with respect to W. Using the Rodrigues' formula (6.12), we have

$$\langle \widehat{P}_n, \widehat{P}_n \rangle = \int \hat{P}_n(t)W(t)t^n dt = \Gamma_n^{-1} \int R_n^{(n)} t^n dt,$$

where Γ_n is the leading coefficient of P_n (6.13). An integration by parts and the formulas for R_n and Γ_n (see Theorem 6.2) then give

$$\langle \widehat{P}_n, \widehat{P}_n \rangle = \frac{\sqrt{\pi}n!}{2^n} \begin{pmatrix} \dfrac{\gamma_{n+1}}{2b^{\frac{2n+1}{2}}} & 0 \\ 0 & \dfrac{2}{\gamma_n} \end{pmatrix}. \tag{6.43}$$

If we write

$$\Delta_n = \sqrt{\frac{2^n}{\sqrt{\pi}n!}} \begin{pmatrix} \sqrt{\dfrac{2b^{\frac{2n+1}{2}}}{\gamma_{n+1}}} & 0 \\ 0 & \sqrt{\dfrac{\gamma_n}{2}}, \end{pmatrix}, \tag{6.44}$$

the polynomials

$$\mathscr{P}_n = \Delta_n \hat{P}_n,$$

are then orthonormal with respect to W.

We now prove that they satisfy the three term recurrence relation (6.16).

This is just a matter of computation. Indeed, the coefficient A_{n+1} in (6.17) is then

$$A_{n+1} = \Delta_n \Delta_{n+1}^{-1}.$$

Formula (6.44) for Δ_n gives now the formula for A_n in (6.17).

On the other hand, for the recurrence coefficient B_n in (6.16) we have the expression $B_n = \Delta_n \hat{B}_n \Delta_n^{-1}$, where

$$\hat{B}_n = \text{coeff. of } t^{n-1} \text{ in } \hat{P}_n - \text{ coeff. of } t^n \text{ in } \hat{P}_{n+1}.$$

From (6.15), we get that

$$\hat{B}_n = (b + (b-1)n) \begin{pmatrix} 0 & \dfrac{a}{2b} \\ \dfrac{2\bar{a}b^{n-\frac{1}{2}}}{\gamma_n \gamma_{n+1}} & 0 \end{pmatrix}, \tag{6.45}$$

and the formula for B_n in (6.17) follows easily.

□

The three term recurrence relation for the polynomials $\{P_n\}$ (6.12) can now easily be computed. Indeed, taking into account the expression for the leading coefficient Γ_n (6.13) of P_n and Δ_n (6.44) of $\mathscr{P}_n$, we find that $P_n = G_n \mathscr{P}_n$ where

$$G_n = \Gamma_n \Delta_n^{-1} = \sqrt{2^n \sqrt{\pi} n!} \begin{pmatrix} \sqrt{\dfrac{\gamma_{n+1}}{2b^{\frac{2n+1}{2}}}} & 0 \\ 0 & \sqrt{2\gamma_n} \end{pmatrix}. \tag{6.46}$$

In particular, this gives $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. If we write

$$tP_n(t) = \tilde{A}_{n+1} P_{n+1}(t) + \tilde{B}_n P_n(t) + \tilde{C}_n P_{n-1}(t), \quad n \geq 0, \tag{6.47}$$

it follows from (6.16) that $\tilde{A}_{n+1} = G_n A_{n+1} G_{n+1}^{-1}$, $\tilde{B}_n = G_n B_n G_n^{-1}$ and $\tilde{C}_n = G_n A_n^* G_{n-1}^{-1}$. An easy computation gives now

$$\tilde{A}_n = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\gamma_{n-1}}{\gamma_n} \end{pmatrix}, \quad \tilde{B}_n = (-n + (n+1)b) \begin{pmatrix} 0 & \dfrac{a}{2b\gamma_n} \\ \dfrac{2\bar{a}b^{n-\frac{1}{2}}}{\gamma_{n+1}} & 0 \end{pmatrix},$$

$$\tilde{C}_n = n \begin{pmatrix} \dfrac{\gamma_{n+1}}{b\gamma_n} & 0 \\ 0 & 1 \end{pmatrix}.$$

The L^2 norm of the polynomials P_n follows easily from the formula (6.46) for the matrices G_n:

$$\langle P_n, P_n \rangle = 2^n \sqrt{\pi} n! \begin{pmatrix} \dfrac{\gamma_{n+1}}{2b^{n+\frac{1}{2}}} & 0 \\ 0 & 2\gamma_n \end{pmatrix}.$$

In a similar way, the three term recurrence relation for the monic orthogonal polynomials $\{\hat{P}_n\}$ can be deduced:

$$t\hat{P}_n(t) = \hat{P}_{n+1}(t) + \hat{B}_n \hat{P}_n(t) + \hat{C}_n \hat{P}_{n-1}(t), \quad n \geq 0,$$

where $\hat{B}_n$ is the one in (6.45) and

$$\hat{C}_n = \frac{n}{2b} \begin{pmatrix} \dfrac{\gamma_{n+1}}{\gamma_n} & 0 \\ 0 & \dfrac{b\gamma_{n-1}}{\gamma_n} \end{pmatrix}.$$

Chapter 7

Conclusions and open problems

In this chapter we summarize the main results of this work and consider some open problems as well.

Chapters 2, 3, and 4 deal with orthogonal polynomials with respect to a linear homogeneous differential operator and asymptotic properties of eigenpolynomials of exactly solvable operators. We have considered the particular cases of Jacobi, Laguerre or Hermite operators and then we generalize some of these results to the case of an exactly solvable operator and a positive Borel measure μ satisfying certain conditions. Chapter 5 is devoted to the study of strong asymptotic properties of eigenpolynomials of linear differential operators and Chapter 6 is devoted to the study of matrix orthogonal polynomials. All the results of the chapters are new and have been submitted for consider for publication in [21, 22, 23, 24, 25].

The necessary and sufficient conditions for uniqueness up to a constant factor of a polynomial of degree n orthogonal with respect to exactly solvable operators are given in Theorem 4.2. This is a difficult problem in general and is still open. As is shown in Example 4.4.1, we can have the existence of operators and measures for which the associated sequence of orthogonal polynomials reduces to a finite set for exactly solvable operators. Theorem 4.3 gives the necessary and sufficient conditions for the existence and uniqueness of an infinite sequence of orthogonal polynomials. We considered also the problem of the existence of sequences of orthogonal polynomials $\{Q_n\}_{n=m}^{\infty}$, where $deg[Q_n] = n$, for some non negative integer m. Theorem 4.4 classifies the operators and the measures for which we can have such sequences. In particular, for either a Jacobi, Laguerre or Hermite operator and assuming that the measure μ is absolutely continuous with respect to the orthogonality measure of the eigenpolynomials of the operator, we show in Theorem 4.5 a relation between the measures μ and the measure of orthogonality of the eigenpolynomials of the operator for the existence of orthogonal polynomials of degree n for $n > m$. We deal with the uniqueness of the sequence in Theorem 4.6 for exactly solvable operators in general.

The study of zero location turns out to be an important matter for the study of asymptotic properties in the theory of orthogonal polynomials. In Theorem 2.6 we show that the zeros of a class of orthogonal polynomials with respect to a Jacobi operator interlace the zeros of Jacobi polynomials and in Corollaries 2.2 and 2.1 we obtain the set of accumulation points for classes of these polynomials. Similar results can be obtained for Laguerre or Hermite operators. Theorem 4.10 extends the classes of measures considered in [12] to obtain the set of accumulation points of the zeros of polar polynomials. Finally, in Theorems 4.8 and 4.9 we obtain a region in the complex plane for orthogonal polynomial with respect to a composition of operators.

The study of asymptotic properties of orthogonal polynomials with respect to a classical operator is done in Theorems 2.2, 2.5 for the case of a Jacobi operator. For the case of Laguerre and Hermite operators is done in Section 3.5. Our main technique in this case consist in the connection of the orthogonal polynomials with respect to the operator and the measure with the eigenpolynomials of the operator and then to use the well known asymptotic properties of these polynomials.

We have considered also the strong asymptotic behavior of eigenpolynomials of exactly solvable operators for the case in which the leading coefficient of the operator is a real polynomial. The main technique consists in using an expansion of the eigenfunctions of the operator in the form of an exponential infinite convergent series which is given in Lemma 5.7. Applying some techniques of complex analysis, functional analysis and supported in the results of the weak convergence of the root counting measure of the zeros given by [15] we give a formula for the strong asymptotic behavior in Theorem 5.2. We show an application in Section 5.4 to polynomials, orthogonal with respect to a class of Sobolev inner product, which are solution of a fourth degree differential equation.

In Sections 2.6 and 3.3 we discuss hydrodynamical models (electrostatic equivalently) for the zeros of these polynomials and the zeros of the derivative as well for the class of measures $\mathcal{P}_1(\alpha,\beta), \mathcal{P}_m[\Delta]$, $m=1$ or $m=2$ defined in Sections 2.1 and 3.2 for the case of Jacobi, Laguerre or Hermite respectively. We also have considered recurrence relations up to a finite number of terms for the derivatives of the polynomials orthogonal with respect to a Jacobi operator and for the case of a Laguerre or Hermite operator we have considered both cases, with and without derivatives.

Chapter 6 concerns with matrix orthogonal polynomials, we introduce a new class of matrix orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ satisfying a second order differential equation with matrix polynomials coefficients. The expression of the weight matrix is given in formula (6.7). Theorem 6.1 classifies the second order differential equations

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0(t) = \Lambda_n P_n(t), \tag{7.1}$$

where Λ_n are Hermitian matrices, for which $\{P_n\}_{n=0}^{\infty}$ is solution. Theorem 6.2 gives a Rodrigues' formula that this class satisfy for the case in which the weight function W is of size $N=2$ and from this Theorem we deduce Corollary 6.1 for an explicit expression of this family. In Theorem 6.3 we give a three term recurrence relation for $N=2$.

7.1 Some open problems

Based upon the results of this work, we consider some open problems for a future research.

1. Let μ be a Borel measure on the real line and $\mathcal{L}^{(M)}$ be a linear homogeneous differential operator satisfying the conditions of Definition (1.4). Find necessary and sufficient conditions for the normality of an index n.

2. Under the same conditions of Item 1, find necessary and sufficient conditions for the existence of a non negative integer m such that the existence of an infinite sequence of orthogonal polynomials with respect to $(\mathcal{L}^{(M)},\mu)$, $\{Q_n\}_{n=m}^{\infty}$ with $deg[Q_n]=n$ can be guaranteed. Analyze the conditions for the uniqueness of the sequence.

3. Let us assume that μ^* is the Jacobi, Laguerre or the Hermite measure and μ a positive Borel measure on $\mathbb{R}$. Consider the Lebesgue decomposition $d\mu(x)=\rho(x)d\mu^*(x)+\sum_{k=1}^{m} c_k\delta_k(x-x_k)$, where $c_1,\ldots,c_k\in\mathbb{R}^+$ are mass points. Classify the function ρ in order to have orthogonal polynomials of degree n with respect to a classical operator and the measure μ , for every $n>m$, for some $m\in\mathbb{N}$.

4. Prove or give a counterexample for the statement that the zeros of the polynomials orthogonal with respect to a classical operator and a measure $\mu=\dfrac{\mu^*}{\rho}$, where ρ is a positive polynomial and μ^* is a Jacobi, Laguerre or Hermite measure are real and interlace with the zeros of the polynomial eigenfunctions of the operator. Prove or give an counterexample also for the case of measures for which the existence of a sequence of orthogonal polynomials with respect to a classical operator of the form $\{Q_n\}_{n=m}^{\infty}$ with $deg[Q_n]=n$ can be guaranteed.

5. Find the accumulation points of the zeros of the polynomials orthogonal with respect to an exactly solvable operator that factorizes on $\mathbb{P}$ (Definition 4.1) and a positive Borel measure μ.

6. Under the same conditions of Item 1, assume additionally that $supp(\mu)$ is bounded, also that there exists a sequence $\{Q_n\}_{n=m}^{\infty}$ of polynomials orthogonal with respect to $(\mathcal{L}^{(M)}, \mu)$ with $deg[Q_n] = n$ for some non negative integer m, and that for each n we have fixed an adequate number of zeros of Q_n on a compact subset $K \subset \mathbb{C}$. Prove or give a counterexample for the statement that the zeros of the sequence $\{Q_n\}_{n=m}^{\infty}$ are uniformly bounded for every $n \in \mathbb{N}$.

7. Find an hydrodynamical model (electrostatic equivalently) for the zeros of polynomials orthogonal with respect to classical operator for a class of measures that were not considered in Sections 2.6 and 3.3 of Chapters 2 and 3 respectively. Consider also the case of higher order operators. Analyze the stability of the equilibrium position.

8. Let μ be a Borel positive measure such that $\mu = \dfrac{\mu^*}{\rho}$, where ρ is a positive polynomial and μ^* is the Laguerre or Hermite measure. Obtain a formula for the strong asymptotic behavior of the orthogonal polynomials with respect to μ and apply this result to obtain a relative asymptotic or the polynomials orthogonal with respect to $(\mathcal{L}, \mu)$, where $\mathcal{L}$ is a Laguerre or Hermite operator.

9. Assume that $\mathcal{L}^{(M)}$ is an exactly solvable operator that factorizes on $\mathbb{P}$ (Definition 4.1) and $\mu \in \Xi_{\mathcal{L}^{(M)}}$ (cf. Section 4.4). Find a logarithmic asymptotic for orthogonal polynomials with respect to $(\mathcal{L}, \mu)$.

10. With the same conditions of Item 9, obtain recurrence relations for orthogonal polynomials with respect to $(\mathcal{L}, \mu)$. Consider also the of the derivatives of these polynomials.

11. Assume that $\mathcal{L}^{(M)}$ is an non degenerate exactly solvable operator. Obtain a formula for the strong asymptotic behavior of the eigenpolynomials of $\mathcal{L}^{(M)}$ for the case in which the leading coefficient of the operator is an arbitrary polynomial. Consider also an expression for the remainder.

12. Assume that $\mathcal{L}^{(M)}$ is a degenerate exactly solvable operator. Obtain a Plancherel–Rotach asymptotic for the strong asymptotic behavior of the eigenpolynomials of $\mathcal{L}^{(M)}$. Consider also a Perron type asymptotic.

13. Obtain a Rodrigues's formula and a three term recurrence relation for the class of matrix polynomials orthogonal with respect to the weight matrix (6.7), for N arbitrary.

14. Find another class of weight matrices for which the matrix orthogonal polynomial with respect to the weight are eigenfunctions of a second-order differential operator having non-diagonal leading coefficient, see Section 1.4.2. Consider also Rodrigues' formulas.

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